SYNCHRONIZATION AND PARAMETER TRACKING IN CHAOTIC SYSTEMS

G. A. JOHNSON, D. J. MAR, T. L. CARROLL AND L. M. PECORA
Code 6343, Naval Research Laboratory, Washington, DC 20375

ABSTRACT

Through optimization techniques, we show that it is possible to combat the problems typically associated with synchronizing experimental systems. Specifically, we concentrate on the effects of parameter mismatch between systems otherwise assumed to be identical. Developing an approach that focuses on conditional eigenvalues, we present results for an experimental system based on the hyperchaotic Rössler equations in addition to a numerical study. In both cases it is shown that the dynamics of the response system can be synchronized to the drive with little error despite substantial mismatch.

1. Introduction

The synchronization of chaotic systems in a drive-response type configuration has been intensely explored since its introduction in 1990.[1] While not exclusively so, the majority of systems studied assume two identical systems, and synchronization is achieved when the differences between corresponding state variables of each system go to zero. Because physical systems cannot be matched perfectly, the minimum conditions that guarantee synchronization in numerical simulations typically fail in experiments.[2] Clearly a robust synchronization approach is necessary for combating the expected experimental hurdles. For example, electronic systems are generally considered low-noise and easy to match, whereas in reality, some components, e.g., capacitors, are difficult to match better than 5%, without resorting to hand-sorting. Additionally, environmental influences can contribute to changes in parameters and drastically alter system dynamics. Of the communication systems that have been developed based on chaos synchronization, many rely on nearly perfect reproduction of the drive variables at the response.[3] Often this is impractical given the expected difficulties. In this paper we address mismatches of inevitable magnitudes and beyond, and develop a strategy to minimize the subsequent synchronization error.

2. Approach

Our synchronization technique is an extension of that recently employed by Peng, *et al.*[4] Its key feature is that the signal that is sent to the response system is a scalar, in the form of a linear combination of the state-space variables, rendering it compatible with existing communication channels. The approach is generalized as follows. Consider identical chaotic systems represented by m-dimensional state vectors $\mathbf{x}(t)$ and $\mathbf{y}(t)$. Given a constant column vector $\mathbf{K}$ of dimension m, the transmitted signal $u(t)$ is formed by the inner product $\mathbf{K}^T\mathbf{x}(t)$. Likewise, $v(t) = \mathbf{K}^T\mathbf{y}(t)$ at the response. A second constant vector $\mathbf{B}$ is constructed and multiplied by the difference $v(t) - u(t)$, then subtracted from the vector field of the response. The dynamics of the complete system is therefore described by

$$d\mathbf{x}(t)/dt = \mathbf{F}(\mathbf{x}(t)) \text{ and } d\mathbf{y}(t)/dt = \mathbf{F}(\mathbf{y}(t)) - \mathbf{B}(v(t) - u(t)). \tag{1}$$

In simulation, the synchronization solution is stable if small perturbations of the response away from $\mathbf{x}(t) = \mathbf{y}(t)$ result only in the system returning to synchrony, i.e., the largest Lyapunov exponent of the response subsystem is negative. Peng, *et al*. show that the problem is reduced to finding an appropriate combination of $\mathbf{B}$ and $\mathbf{K}$ vectors, and that indeed carefully chosen vectors will produce synchronization - even when the drive system itself has multiple positive Lyapunov exponents.

The questions remain: How does one find the right **BK** set for a given vector field **F**, and how can we make it robust enough for real systems? In the next few sections, we will develop the framework of a straightforward strategy that makes it possible to find multiple appropriate combinations, and concurrently maximize the performance of the synchronized system in the presence of substantial parameter mismatch.

3. Piecewise-linear 4-dimensional Rössler System

Our experimental system is modeled on the hyperchaotic Rössler equations[5]. The system is an analog computer circuit with four integrating amplifiers and two piecewise linear components provided by diodes. In a typical regime, the system is closely modeled by

$$\begin{aligned}
dx_1/dt &= -.05x_1 - .5x_2 - .62x_3 \\
dx_2/dt &= x_1 + \rho x_2 + .40x_4 \\
dx_3/dt &= -2x_3 + g(x_1) \\
dx_4/dt &= -1.5x_3 + .18x_4 + h(x_4) \\
g(x_1) &= 10(x_1 - .68)\,\Theta(x_1 - .68) \\
h(x_4) &= -.41(x_4 - 3.8)\,\Theta(x_4 - 3.8)
\end{aligned} \tag{2}$$

where ρ is a convenient bifurcation parameter usually varied between .05 and .12 and $\Theta(.)$ is the Heaviside step function (1 for positive argument, 0 for negative).

3.1. Synchronization Criteria and Optimization

If we express the drive system dynamics as $d\mathbf{x}(t)/dt = \mathbf{A}(t)\mathbf{x}(t)$, where $\mathbf{A}$ is a piecewise constant matrix, the receiver subsystem is then given by $d\mathbf{y}(t)/dt = \mathbf{A}(t)\mathbf{y}(t) - \mathbf{B}\mathbf{K}^T(\mathbf{y}(t) - \mathbf{x}(t))$. In linear control theory, if $\mathbf{A}$ is a constant matrix, the subsystem is considered stable if the real parts of the eigenvalues, or Re[λ], of [$\mathbf{A} - \mathbf{B}\mathbf{K}^T$] are negative. Since the stability condition is also the synchronization condition, $\mathbf{x}(t) = \mathbf{y}(t)$, this forms the basis for our optimization approach. Because our system matrix $\mathbf{A}(t)$ switches between four constant matrices, we require that a chosen **BK** combination results in negative Re[λ] of the four possible subsystem matrices [$\mathbf{A}_n - \mathbf{B}\mathbf{K}^T$] ($n$=1..4). That is to say, we treat each of the four possible states of the system as a separate control problem. If each subsystem is stable, we expect that the system taken as a whole is likewise stable. To ensure robust synchronization, we take this approach one step further and demand that the magnitudes of the negative Re[λ] be as large as we can make them.

To find the most desirable eigenvalues, it is straightforward to write a routine that executes the general approach as follows. Starting from a random point in BK-space, we calculate the largest Re[λ] in each of the four subsystem matrices. The value of each largest eigenvalue is weighted by the percentage of time the system typically spends in each region, and their sum is the quantity we minimize. From the random initial point, our algorithm seeks out the local minimum of the sum by moving about in BK-space Typically, we distribute a uniform grid of starting points in the space so that the best local minima are located. Also, we restrict the magnitude of the **B** and **K** components in order to remain physically realistic.

3.2. Experimental Results

For the data presented here from the piecewise linear 4d Rössler circuit, we choose **K** = (-1.97, 2.28, 0, 1.43) and **B** = (.365, 2.04, -1.96, 0) from hundreds of possible similar local minima. In compliance with our criteria, the resulting largest eigenvalues in the four response matrices, $[\mathbf{A}_n - \mathbf{BK}^T]$, are (-1.4, -.96, -.50, -.16). While we find that the ensuing synchronization is both rapid and robust, for the purposes of this paper, we concentrate on the strong resistance to parameter mismatch.

To demonstrate the mismatch tolerance, we vary the bifurcation parameter ρ of the drive system while holding the same response parameter constant. From Fig. 1 it is evident that the synchronization of the response to the drive is not adversely affected even as the drive system is drawn through its entire range of dynamical behavior, from a period-1 limit cycle into a regime of chaos with two positive Lyapunov exponents. The figure shows the x_2 vs x_1 view of both drive and response systems, while the bar graphs indicate the magnitudes of the ρ parameter of each system. Despite parameter differences of 50% or more, the response system reproduces the dynamics of the drive without significant distortion - as though both parameters were being varied together.

4. Extension to Nonlinear Systems

The piecewise-linear nature of our experimental system represents a fortunate circumstance for calculating stability of the synchronization solution, as there are a finite number of possible matrices to evaluate. However, for nonlinear systems where $\mathbf{A}(t)$ is a continuously changing matrix, our optimization strategy needs to be revised. A control theory approach to dealing with explicit time dependence in **A** is to determine the stability of the response matrix $[\mathbf{A}(t) - \mathbf{BK}^T]$ at several frozen points in time - the so-called frozen coefficient method.[6] Such a criteria was proposed for synchronization of chaotic systems, that is, Re[λ] of the Jacobian are negative pointwise around the attractor.[7] For our purposes, we couple this technique to our optimization routine and show that we can maximize parameter mismatch tolerance in truly nonlinear systems.

4.1. Frozen Variable Method - Lorenz

Our autonomous nonlinear systems have time-dependence built into the variables, so the traditional approach is slightly modified. The well-known Lorenz system, for example, has a system matrix $\mathbf{A}(\mathbf{x}(t))$ that depends explicitly on $x_1(t)$, $x_2(t)$ and $x_3(t)$ rather than time,

$$\mathbf{A}(\mathbf{x}(t)) = \begin{pmatrix} -16 & 16 & 0 \\ R - x_3(t) & -1 & -x_1(t) \\ x_2(t) & x_1(t) & -4 \end{pmatrix} \tag{3}$$

where $d\mathbf{x}(t)/dt = \mathbf{A}(\mathbf{x}(t))\,\mathbf{x}(t)$. To evaluate the stability of the response for a set of **B** and **K** vectors, we first sample x_1, x_2 and x_3 at 1000 points on the attractor at random or regular intervals in time. These values are in turn used to calculate the "frozen" **A** matrix at each of the 1000 points. The eigenvalues of $[\mathbf{A}_n - \mathbf{BK}^T]$ (n = 1 to 1000) are determined, and we require that all Re[λ] are negative for each matrix. If this condition is met, we then optimize the system by minimizing Re$[\lambda]_{max}$ taken as an average over the 1000 points, retaining the constraint that they are negative everywhere. As in the case of the piecewise-linear system, we identify hundreds of **BK** combinations this way that satisfy our criteria and result in robust synchronization between identical systems. Note that 1000 points are taken to ensure a representative sampling of the attractor. Our experience suggests that, at least for Lorenz, this number can be greatly reduced.

To evaluate the effects of parameter mismatch in the Lorenz system, we vary the R_X parameter at the drive system. In Fig. 2, we display the x_2 vs. x_1 attractors at both response and drive for the situation in which the response parameter R_y is fixed at 200 while the drive parameter takes on the values 120, 165, 235 and 334 corresponding to chaotic regimes, a periodic window, and a periodic (P4) regime. As shown, the drive dynamics are well reproduced at the receiver despite relative parameter mismatches of more than 60%. For a less qualitative assessment, we now quantify and estimate a practical bound for the synchronization errors for a given mismatch.

4.2 Error Estimation

Let the synchronization error $\xi(t)$ be taken as the vector difference between the drive and response, $\mathbf{x}(t)$ - $\mathbf{y}(t)$. Then the linearized expression for the time evolution of the error can be shown to be

$$d\xi(t)/dt = [\mathbf{A}(\mathbf{y}(t),R_y) - \mathbf{BK}^T]\,\xi(t) + \delta(t) \tag{4}$$

where $\delta(t)$ is a source term due to parameter mismatch, $\delta(t) = (R_X - R_y)[\ \mathbf{F}(\mathbf{x}(t),R_X)/\ R_X]$. An expression of the long-term solution for the norm of the error can be shown to be

$$\|\xi(t)\| \le \int_{-\infty}^{t} \|\mathbf{\Phi}(t,\tau)\|\,\|\boldsymbol{\delta}(\tau)\|\,d\tau \quad \le \quad \|\boldsymbol{\delta}(t)\|_{\max} \int_{-\infty}^{t} \|\mathbf{\Phi}(t,\tau)\|\ d\tau. \tag{5}$$

Further estimating the principle matrix norm, $\|\Phi(t,\tau)\| \le \exp(\Lambda(t-\tau))$, where Λ is the largest Lyapunov exponent of the response subsystem, and integrating gives

$$\|\xi(t)\|_{max} < \|\delta\|_{max}\,(-\Lambda)^{-1} \tag{6}$$

which similar arguments also reveal.[8, 9]

We test this bound estimate by taking a thousand optimized **BK** combinations, and for each set, integrating drive and response equations with the parameters mismatched ($R_X = 165$, $R_y = 200$). The maximum and average norm errors are recorded for each set and plotted against the largest Lyapunov exponent in Fig. 3. Also plotted are the maximum error estimates for each **BK** set, where all errors are expressed as a fraction of the maximum norm of the attractor, $\|\mathbf{x}\|_{max}$. While the average error is shown to be fairly consistent regardless of λ, it is clear that the maximum excursions away from synchrony are effectively suppressed as λ becomes increasingly negative.

5. Concluding Remarks

Because commercial products based on synchronized chaotic systems appear to be on the horizon, it is important that such systems can be constructed to be robust against (among other negative effects) parameter mismatch. We have shown that optimization of the coupling components *already in place* (i.e., no added parts or labor) can result in impressive resistance to differences in systems that are otherwise assumed to be identical. Optimization strategies rooted in linear control theory were put forth for both piecewise-linear and continuously nonlinear systems using only scalar transmitted signals. A heuristic estimation of the error was introduced and shown to be reasonably effective in predicting maximum bursts away from the synchronization manifold.

6. References

1. L. M. Pecora and T. L. Carroll, *Phys. Rev. Lett.* **64** (1990) 821; *Phys. Rev.* **A44** (1991) 2374.
2. . D. Gauthier, in *Proc. 4th Exp. Chaos Conf*. (World Scientific, Singapore, 1998).
3. See for example Ned J. Corron, in *Proc. 4th Exp. Chaos Conf.* (World Scientific, Singapore, 1998).
4. J. H. Peng, E. J. Ding, M. Ding and W. Yang, *Phys. Rev. Lett.* **76** (1996) 904.
5. O. E. Rössler, *Phys. Lett.* **A71** (1979) 155.
6. W. L. Brogan, *Modern Control Theory* (Prentice Hall, New Jersey, 1991) p. 359.
7. L. Pecora, T. Carroll, and J. Heagy, *Chaotic Circuits for Communications, Photonics East,,SPIE Proceedings* (SPIE, Bellingham, WA, 1995) 25.
8. R. Brown, N. Rulkov and N. B. Tufillaro, *Phys. Rev.* **E50** (1994) 4488.
9. M. A. Matías and J. Güémez, *Phys. Lett.* **A209** (1995) 48.

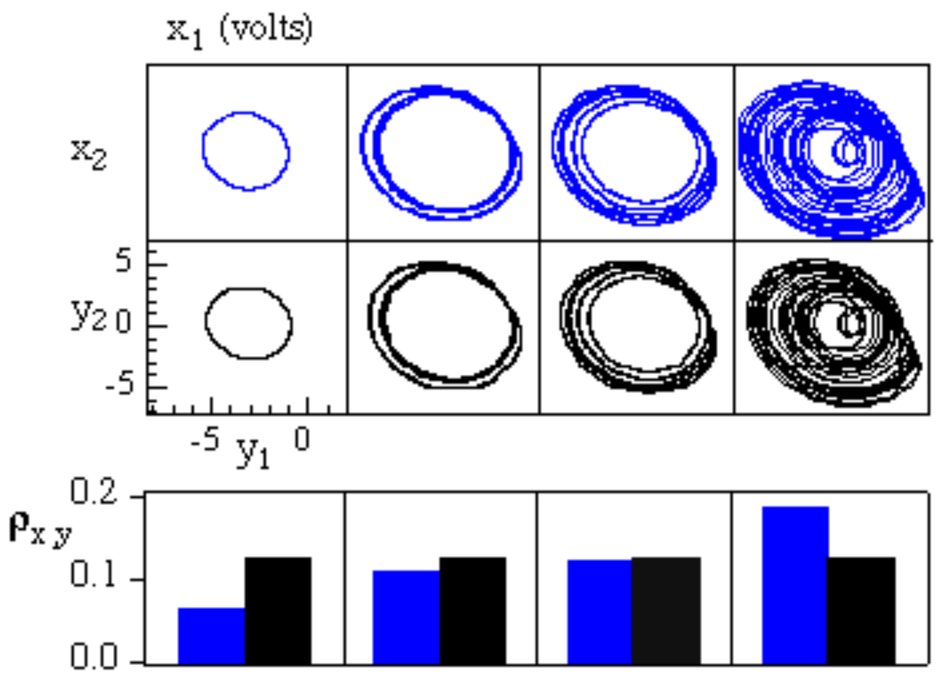

Fig. 1. The dynamics of the response circuit, **y**, are shown to match the drive dynamics even as the response parameter (the darker bar) is held constant while the drive parameter varies over a wide range. Each square in the upper figure has axis limits shown on the bottom left square. The sets of bar graphs correspond to the attractors directly above each.

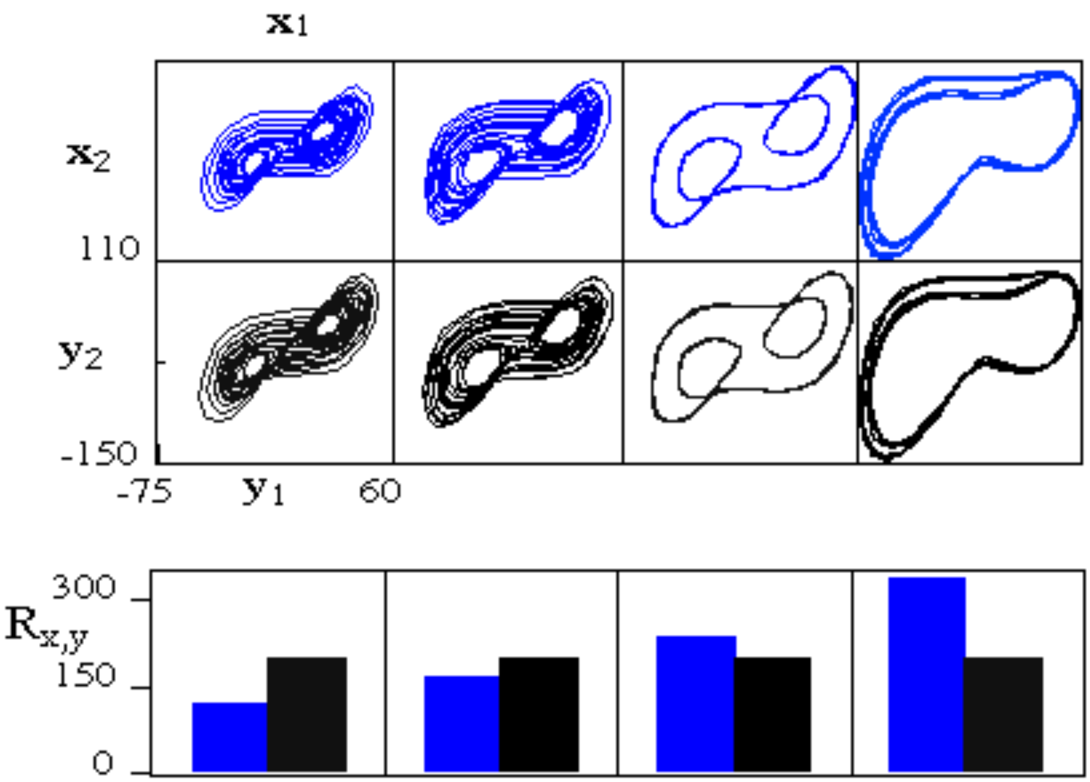

Fig. 2. Analogous to Fig. 1, the dynamics of the response system **y** are shown to follow the drive as only the drive parameter (the lighter bar) is varied. The axes in each plot have the same limits as shown in the bottom left.

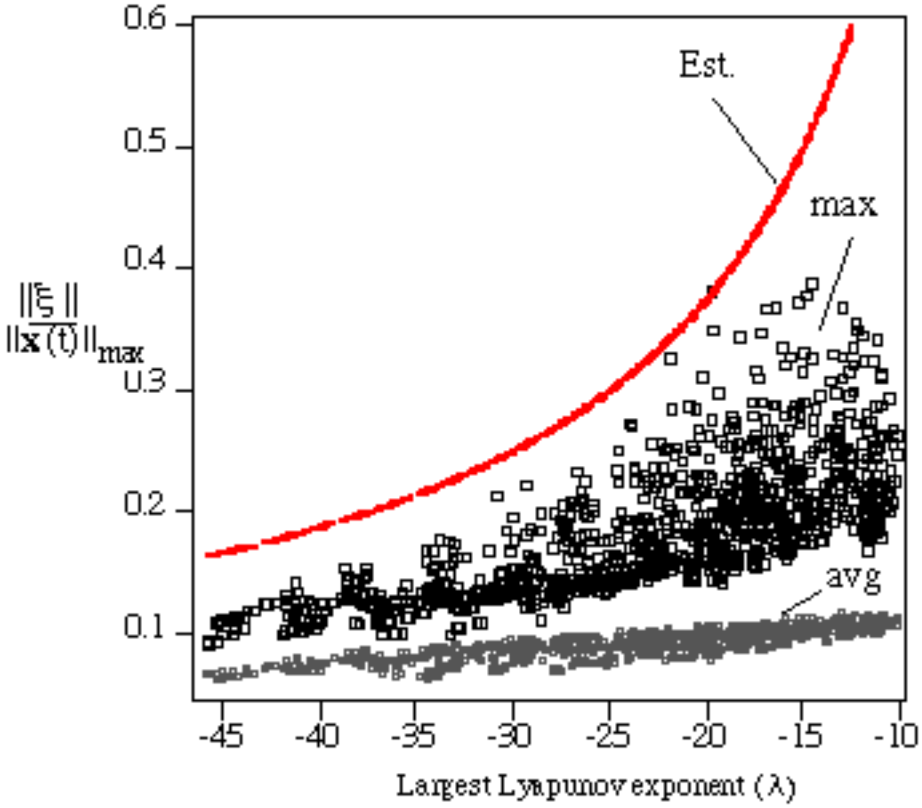

Fig. 3. For coupled Lorenz systems the maximum, average and estimated errors are plotted as a fraction of the attractor size for 1000 **BK** combinations with R_X and R_Y set to 165 and 200. For largely negative Lyapunov exponents, the maximum errors are clearly suppressed more effectively than for moderate values.