SYNCHRONIZATION AND IMPOSED BIFURCATIONS IN THE PRESENCE OF LARGE PARAMETER MISMATCH

G. A. Johnson, D. J. Mar, T. L. Carroll, and L. M. Pecora

Code 6343, U. S. Naval Research Laboratory

Washington, DC 20375

Using experimental and numerical systems, we show that it is possible to maintain excellent synchronization between a drive and response system even when there is large (50%) parameter mismatch between them and they are coupled only though a scalar signal. By optimizing the coupling consistent with a stability constraint, we show that a consequence of the optimized coupling is that the synchronization is maintained even in the presence of bifurcations in the drive system - despite the condition the response parameters are held constant.

To appear in Physics Review Letters, 1998

The curious ability of two systems to exhibit unpredictable, chaotic behavior yet evolve in perfect synchrony has generated great interest, mostly because of potential applications in a communications capacity [1-5]. In such a role, one typically has a unidirectionally coupled pair of identical systems with a response or receiver system that synchronizes to the dynamics of the drive or transmitter system. If the dynamics of the drive contains an information signal, the ability to recover the information at the response hinges on the ability to faithfully reproduce the drive dynamics. Depending on the coupling technique, the problem is easily complicated by system mismatch, as in some cases component differences of a few percent have been shown to be sufficient to induce momentary large bursts away from synchronization [6-8], seriously limiting the use of such systems. In this Letter we show that by optimizing a coupling configuration borrowed from control theory, we can achieve such a highly robust synchronization with a single scalar transmitted signal that the response can closely reproduce the drive dynamics despite parameter mismatches well beyond expected levels, e.g., due to component tolerances or environmental influences. Because changes in a parameter can result in significant changes in dynamics, this robust synchronization and its immunity to parameter mismatches allows the dynamics of the response to be varied through bifurcations and different dynamical regimes solely by varying a parameter at the *drive* end.

While the necessary and sufficient stability conditions of the synchronized state are a current topic of debate [6, 9-11], we focus on local stability and optimize our approach to guarantee robust synchronization. Our criterion, generally stated, is as follows. Given a chaotic system, $\mathbf{x}(t)$, where $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, and a second system, $\mathbf{y}(t)$, governed by $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y},\mathbf{x})$, we require that the eigenvalues of the response Jacobian, $\mathbf{A}=\mathrm{D}\mathbf{G}$, have negative real parts everywhere on the attractor. For different systems, the mechanics of satisfying this constraint vary somewhat. For piecewise-linear systems, where the system switches between a finite number of constant Jacobians, we simply ensure that our coupling renders the real parts of the eigenvalues negative for each Jacobian. For smoothly nonlinear systems (not *piecewise*-linear) we choose the more exhaustive approach of ensuring that the instantaneous eigenvalues of the time-varying response

Jacobian, $\mathbf{A}(t)$, have negative real parts everywhere on the attractor. Other criteria of varying stringency for synchronization [12] could be similarly realized with our coupling configuration and accompanying optimization routine, yet we have found that optimization of our criterion is sufficient to provide excellent synchronization in experimental and numerical examples, even under conditions of substantially mismatched parameters and noise.

Our choice of coupling configuration was motivated by the recent results of Peng *et al.* [13] in which it was shown that systems with more than one positive Lyapunov exponent can be synchronized with a scalar transmitted signal. A key feature of the technique is that it provides $2m$ adjustable coupling parameters for m-dimensional systems. The strategy may be generally outlined by the following. To generate the transmission signal u, we define $u(t) = \mathbf{K}^T\mathbf{x}(t)$ where $\mathbf{K}$ is a constant column vector, $(k_1, k_2, \ldots, k_m)^T$, and $\mathbf{x}(t)$, the drive system state vector. Likewise, the response state vector $\mathbf{y}(t)$ is used to generate a second scalar signal, $v(t) = \mathbf{K}^T\mathbf{y}(t)$. The difference between the two scalars is multiplied by a second constant vector, $\mathbf{B}$, and subtracted directly from the vector field, $\mathbf{F}$, of the response subsystem. Otherwise stated,

$$d\mathbf{y}(t)/dt = \mathbf{F}(\mathbf{y}(t)) - \mathbf{B}\mathbf{K}^T(\mathbf{y}(t) - \mathbf{x}(t)) \qquad (1)$$

and $\mathbf{A}$ becomes $[D\mathbf{F} - \mathbf{B}\mathbf{K}^T]$. This configuration is a general form of linear coupling with a transmitted scalar, and its relation to a coordinate transformation is addressed in Ref. [14].

Introducing a parameter mismatch, $\Delta\mu = \mu_x - \mu_y$, so that the vector fields of the two systems are no longer identical, we define the synchronization error as $\xi(t) = \mathbf{x}(t) - \mathbf{y}(t)$. Then

$$d\xi(t)/dt = d\mathbf{x}(t)/dt - d\mathbf{y}(t)/dt = \mathbf{F}(\mathbf{x}(t), \mu_x) - \mathbf{F}(\mathbf{y}(t), \mu_y) + \mathbf{B}\mathbf{K}^T(\mathbf{y}(t) - \mathbf{x}(t)). \qquad (2)$$

Rewriting and linearizing about the synchronized state $\mathbf{y} = \mathbf{x}$ gives

$$d\xi(t)/dt = [\mathbf{J}(\mu_y, t) - \mathbf{B}\mathbf{K}^T]\xi(t) + \delta(t) \qquad (3)$$

where $\mathbf{J}(\mu_y) = [\partial\mathbf{F}(\mathbf{y}(t), \mu_y)/\partial\mathbf{y}]_{\mathbf{y}=\mathbf{x}}$ and $\delta(t)$ is a source term due to the parameter mismatch, $\delta(t) = \Delta\mu\,[\partial\mathbf{F}(\mathbf{y}(t),\mu_y)/\partial\mu_y]$. For identical systems, the source term vanishes and synchronization of the two chaotic systems is realized if the norm $||\xi(t)||$ approaches 0 in the long time limit. Without the luxury of identical systems - a typical experimental condition - the level of synchronization is limited by the existence of the source term. Solving for the error between mismatched systems gives

$$\xi(t) = \Phi(t)\Phi^{-1}(t_o)\xi_o + \int_{t_o}^{t} \Phi(t)\Phi^{-1}(\tau)\delta(\tau)\, d\tau \qquad (4)$$

where $\Phi(t)$ is the principle matrix function defined by $d\Phi(t)/dt = \mathbf{C}(t)\Phi(t)$ where $\mathbf{C}(t) = [\mathbf{J}(\mu_y, t) - \mathbf{BK}^T]$[15]. Taking norms in the limit of $t_o = -\infty$, we establish the inequality

$$\|\xi(t)\| \leq \int_{-\infty}^{t} \|\Phi(t,\tau)\| \, \|\delta(\tau)\| \, d\tau \qquad (5)$$

where the ξ_o term in Eq. (4) goes to zero provided the Lyapunov exponents are negative. Note we have taken $\Phi(t,\tau)$ to be short for the product $\Phi(t)\Phi^{-1}(\tau)$. Assuming this to be the case, we establish an upper bound on the error norm, $\|\xi(t)\|_{max}$, by taking the maximum magnitude of the source term, $\|\delta(t)\|_{max}=|\Delta\mu|\|\partial\mathbf{F}(\mathbf{y}(t),\mu_y)/\partial\mu_y\|_{max}$ then integrating over $\|\Phi(t,\tau)\|$. Taking $\|\Phi(t,\tau)\| \leq \exp(\Lambda(t-\tau))$, where Λ is the largest instantaneous Lyapunov exponent of the response subsystem, we estimate the integral to be λ^{-1} where λ is the average largest Lyapunov exponent as measured at a representative sampling around the attractor. We take the average exponent in part because of the averaging effect of the integration over rapidly changing exponents, and, to be shown later, because it matches up well with our optimization scheme. Then the maximum error magnitude between two mismatched systems is given by

$$\|\xi(t)\|_{max} \leq \|\delta(t)\|_{max}(-\lambda)^{-1} \qquad (6)$$

which alternative arguments have also revealed [16, 17].

The problem of mismatched systems suggests profit in utilizing an optimization algorithm for the following reasons: (*i*) there are 2*m* adjustable parameters in our coupling scheme - too many for simple trial and error searches for the best combinations, and (*ii*) the result stating that the maximum error varies inversely with the largest conditional Lyapunov exponent clearly implies that the performance will be maximized when the largest exponents are as negative as possible. Note that this implies that despite potentially large parameter mismatch between drive and response systems (contained in δ), the drive dynamics can be closely reproduced at the response provided we are able to find a **B-K** set which adequately minimizes negative exponents of the response subsystem. The following two examples serve to outline our optimization approach for two different types of chaotic systems, and to demonstrate both the ability of the response to reproduce and track the dynamics of the drive and the dependence of the maximum error on the conditional Lyapunov exponents.

Example 1 - Our experimental system is a four-dimensional piecewise-linear electronic system that is modeled after the hyperchaotic Rössler equations [18]. In a typical regime, the system is closely modeled by

$$dx_1/dt = -.05x_1 - .5x_2 - .62x_3 \tag{7}$$

$$dx_2/dt = x_1 + \rho x_2 + .40x_4 \tag{8}$$

$$dx_3/dt = -2x_3 + g(x_1) \tag{9}$$

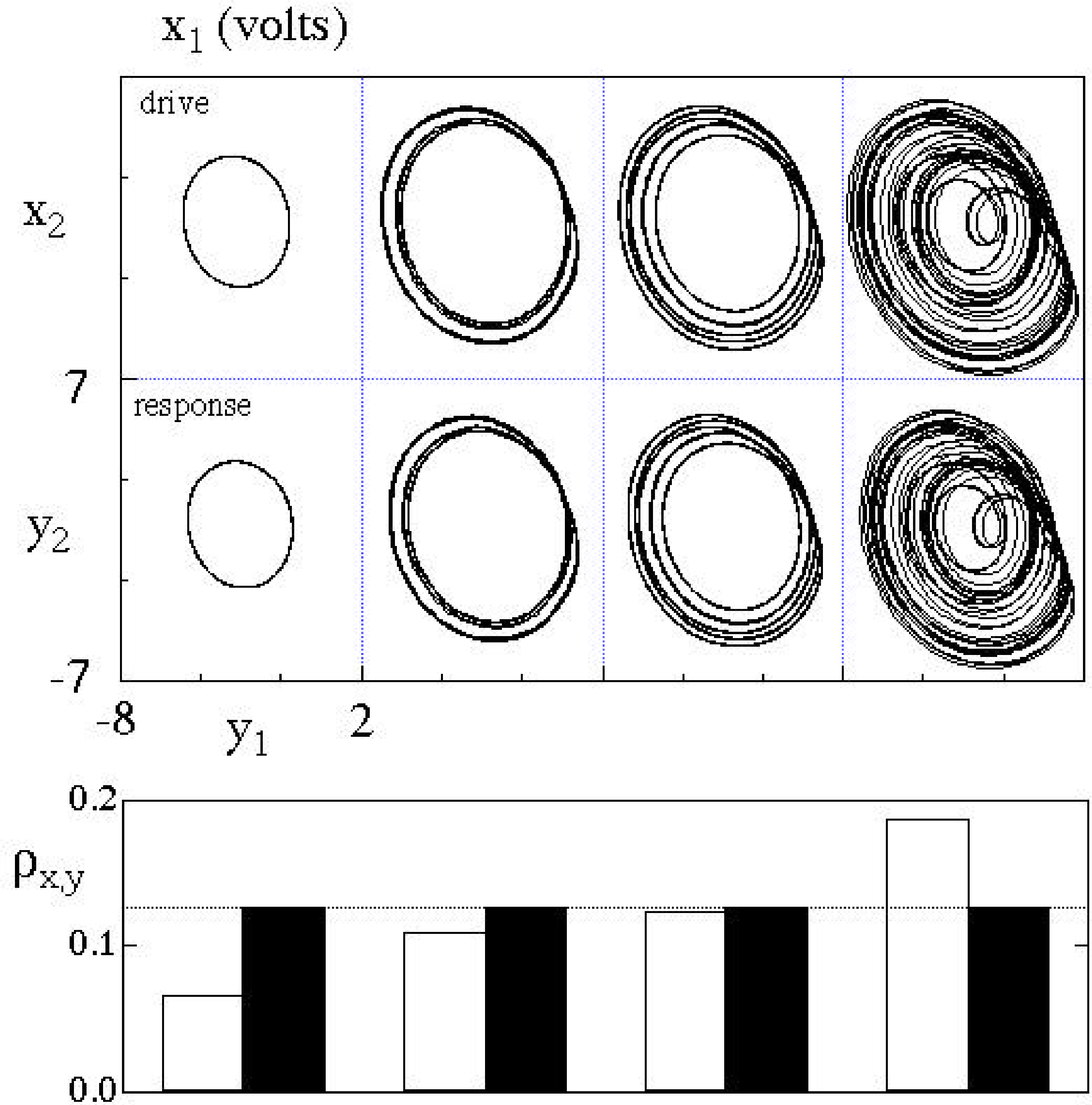
$$dx_4/dt = -1.5x_3 + .18x_4 + h(x_4) \tag{10}$$

$$g(x_1) = 10(x_1 - .68)\,\Theta(x_1 - .68) \tag{11}$$

$$h(x_4) = -.41(x_4 - 3.8)\,\Theta(x_4 - 3.8) \tag{12}$$

where ρ is a convenient bifurcation parameter usually varied between .05 and .19 and $\Theta(.)$ is the Heaviside step function. At a given time, the dynamics of the system is governed by one of four constant response Jacobians, $[\mathbf{J}_n - \mathbf{B}\mathbf{K}^T]$ ($n = 1,...,4$), where $\mathbf{J}_n$ is the normal Jacobian of the equations and the $\mathbf{B}\mathbf{K}^T$ term accounts for the coupling. To optimize the coupling parameters in the piecewise-linear case, we consider the system as four separate linear control problems and focus on the real parts of the eigenvalues of each Jacobian. The criterion demands that the largest real parts of the eigenvalues of each Jacobian are negative, and we optimize the system by minimizing a weighted sum of the real parts, weighted by the amount of time typically spent in each of the four regimes. The sum is optimized by a numerical routine [19] which searches the eight-dimensional **B**-**K** space for local minima of this value. In our search, it is interesting to note that while we intentionally restrict the magnitudes of the *b*s and *k*s to be roughly on the order of the coefficients in the vector field, allowing for arbitrarily large values typically provides little improvement in stability. Thus the optimal coupling parameters are not simply provided by the largest coupling strengths obtainable, rather, a more subtle approach is in fact required to find the best values. The best local minima in our restricted space are recorded, and typically we find that up to several hundred different minima scattered throughout the **B**-**K** space can be located that have similar magnitudes. For the data presented here from the piecewise-linear 4-d circuit, we choose **K** = (-1.97, 2.28, 0, 1.43) and **B** = (.365, 2.04, -1.96, 0). In compliance with our criterion, the resulting largest real parts of the eigenvalues in the four response matrices are (-1.4, -.96, -.50, -.16).

Applying the numerical optimization results to the circuits proves to provide rapid and robust synchronization. For this and other sets of coupling parameters, the synchronization between systems is achieved down to the noise level in a quarter of the period of the natural oscillation frequency of the 4-d circuit (about 200 microseconds). In Fig. 1 we show that the synchronization is maintained between the two systems in all regimes of the drive circuit. We stress that the bifurcation parameter of the response subsystem remains constant, and *only* the drive system parameter is varied to produce the changes in dynamical behavior of *both* systems. From the views of the attractors pictured, a qualitative match between the dynamics of the two systems is evident as the drive system is varied from a period-1 limit cycle, through bifurcations, and into regimes of chaos and hyperchaos. We refer to the bifurcations of the response system as *imposed bifurcations*. The bar graphs indicate the parameter values of each system, the dotted line indicating that ρ_y is held fixed. The maximum mismatch shown is 52%.



x1 (volts)
drive
x2
7
response
y2
-7
-8
y1
2
0.2
ρx,y
0.1
0.0

Example 2 - Our approach can be extended to also encompass nonlinear systems with continuously varying Jacobians, and we take as an example the well-known Lorenz equations

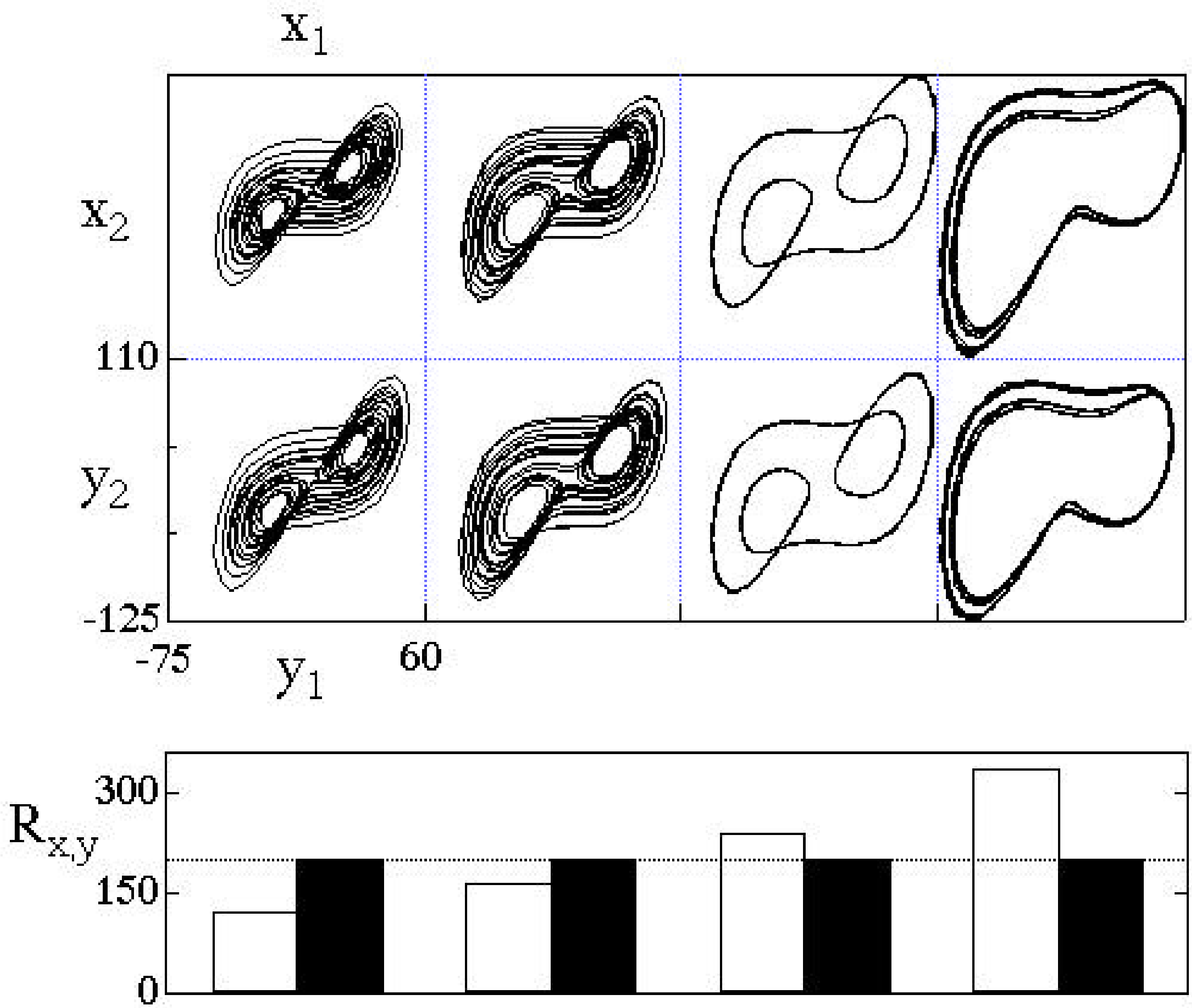
$$dx_1 / dt = \sigma (x_2 - x_1) \tag{13}$$

$$dx_2 / dt = x_1 (R - x_3) - x_2 \tag{14}$$

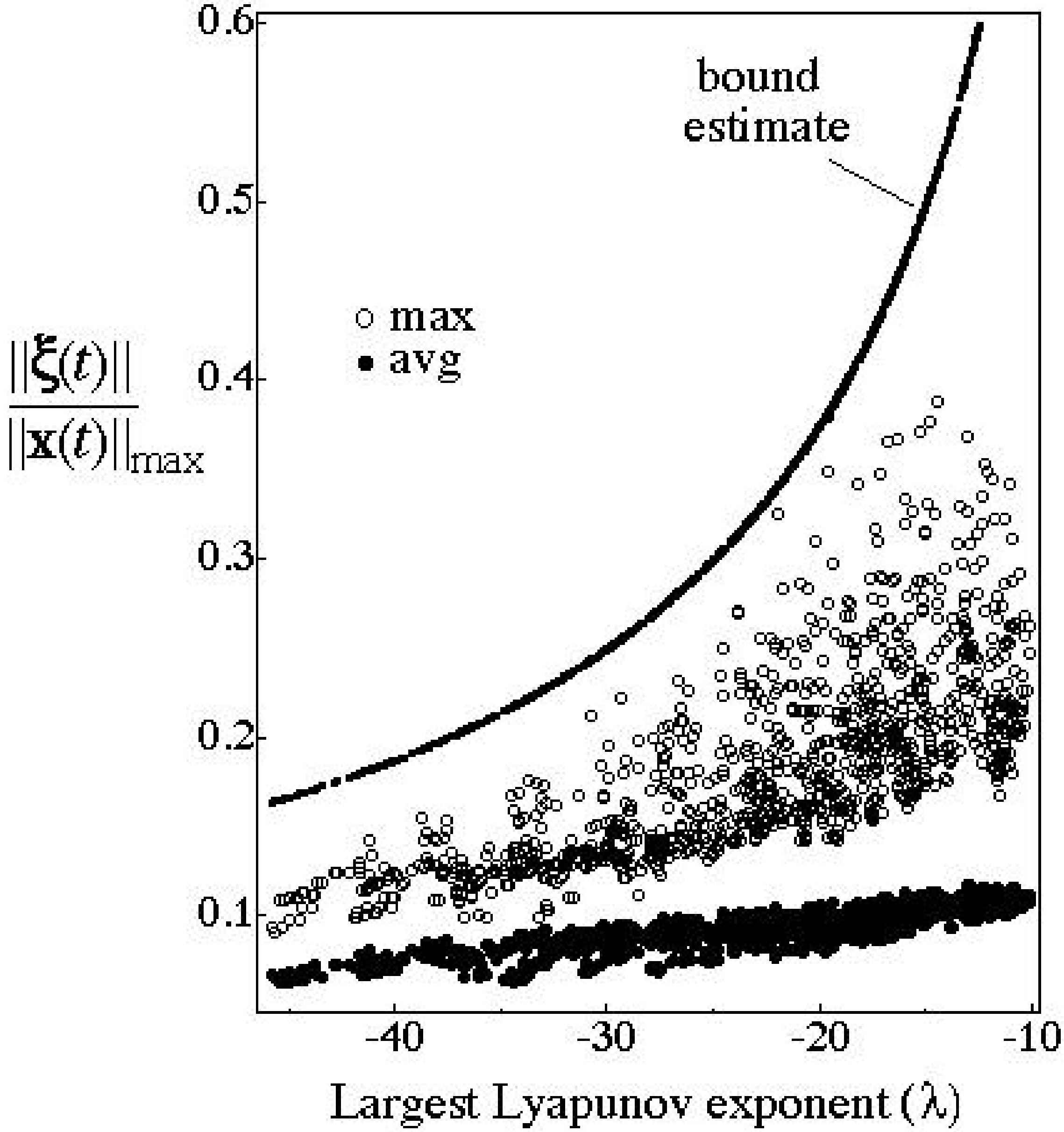
$$dx_3 / dt = x_1 x_2 - b x_3 . \tag{15}$$

Similar to the frozen-coefficient method in control theory [20], we evaluate the eigenvalues of the response Jacobian at a number of samplings of $\mathbf{y}(t)$ around the attractor. Such a criterion was previously proposed for synchronization of chaotic systems, that is, the real parts of the eigenvalues of the response Jacobian are required to be negative pointwise around the attractor [10].

To maximize the robustness of the synchronization between mismatched Lorenz systems with the **BK**-coupling approach, we optimize the response system in the following way. One thousand points separated by a fixed time, Δt, are sampled on the Lorenz attractor with $R=R_y$. At each point, the real parts of the eigenvalues (which we abbreviate as Re[ζ_n]) of $[\mathbf{J}(t) - \mathbf{BK}^{\mathrm{T}}]_{t=n\Delta t}$ (n=1, ..., 1000) are evaluated. We demand that Re[ζ_n] are negative for all n and the system is optimized by minimizing the mean largest Re[ζ_n] as averaged over the thousand points. Again this is achieved by our optimization algorithm probing the **BK**-space in search of local minima of the mean, while retaining the constraint that all of the real parts of the eigenvalues are negative everywhere.



As in the piecewise-linear case, we find that hundreds of local minima exist in **BK**-space, and all result in rapid, robust synchronization between identical Lorenz systems. In the presence of significant parameter mismatch, we find that the response system dynamics closely follows that of the drive. Similar to Fig. 1, in Fig. 2 we show the attractors of the drive and response at different parameter settings of the drive. The response parameter, R_y, remains constant at 200, and the drive parameter, R_x, is set to 120, 165, 235, and 334. The bar graphs in the figure indicate the parameter setting for the attractors just above them. Despite the large mismatch in parameter and subsequent changes in dynamics, the response system is quite successful in reproducing the drive dynamics with little perceivable distortion, and again we see imposed bifurcations in the response system.



To test our estimate for the maximum synchronization error between two mismatched systems, $||\xi(t)||_{max}$, we integrate two Lorenz systems with 1,100 different sets of **B** and **K** vectors corresponding to local minima as described above with $R_X = 165$ and $R_Y = 200$. Ignoring initial transients, for each set the magnitudes of both the average and maximum errors are recorded and plotted in Fig. 3 against the largest conditional Lyapunov exponent, Λ. For each set, the estimated maximum is also plotted, as calculated from Eq. (7), and all three quantities are expressed as a fraction of the size of the attractor, $||\mathbf{x}||_{max}$. While the average error is shown to decrease gradually as Λ becomes increasingly negative, the maximum error is clearly strongly dependent on the exponent and is effectively suppressed for large negative values. Moreover, for the range of Λ plotted, the estimated maximum error is shown to be a quite reasonable upper bound to the measured errors.

Although both the experimental and numerical examples demonstrate intolerance to a parameter shift in a linear term of the vector field, the results are similar for shifts in the nonlinear parameters as well. In the Lorenz system, for example, we can vary the multiplier in the $x_1 x_2$ term in Eq. (15) so that the drive and response are mismatched up to 30% while the induced synchronization error $||\xi(t)||/||\mathbf{x}||_{max}$ remains below .05.

In summary, we have described straightforward criteria for enhancing the synchronization between systems that are not identical with the transmission of a scalar signal. For both nonlinear and piecewise-linear systems, we put forth strategies for minimizing the relevant quantities in each system with an intelligent search through coupling-parameter space. Subsequently, we found that optimization of the coupling allows the response system to reproduce the dynamics of the drive system, even as the drive drags the response through bifurcations and into regimes of chaos and hyperchaos.

References

[1] L. M. Pecora and T. L. Carroll, Physical Review Letters **64**, 821 (1990).

[2] K. Cuomo and A. V. Oppenheim, Physical Review Letters **71**, 65 (1993).

[3] C. W. Wu and L. O. Chua, International Journal of Bifurcations and Chaos **3**, 1619-1627 (1993).

[4] N. Rulkov, M. M. Sushchik, L. S. Tsimring and H. D. I. Abarbanel, Physical Review **E 51**, 980 (1995).

[5] L. Kocarev and U. Parlitz, Physical Review Letters **74**, 5028 (1995).

[6] D. J. Gauthier and J. C. Bienfang, Physical Review Letters **77**, 1751 (1996).

[7] P. Ashwin, J. Buescu and I. Stewart, Physics Letters **A 193**, 126-139 (1994).

[8] N. F. Rulkov and M. M. Sushchik, International Journal of Bifurcation and Chaos **7**, 625 (1997).

[9] R. Brown and N. F. Rulkov, Physical Review Letters **78**, 4189 (1997).

[10] L. Pecora, T. Carroll and J. Heagy, in Chaotic Circuits for Communications, Photonics East, SPIE, (SPIE, 1995), 25-36.

[11] T. Kapitaniak, International Journal of Bifurcations and Chaos **6**, 211 (1996).

[12] For example, we could require that the largest Lyapunov exponent be negative everywhere, or that the eigenvalues of $[\mathbf{A} + \mathbf{A}^{T}]$ (see [6]) are negative everywhere.

[13] J. H. Peng, E. J. Ding, M. Ding and W. Yang, Physical Review Letters **76**, 904-907 (1996).

[14] T. L. Carroll, J. F. Heagy and L. M. Pecora, Physical Review **E 54**, 4676 (1996).

[16] M. A. Matías and J. Güémez, Physics Letters A **209**, 48-52 (1995).

[15] L. Perko, *Differential Equations and Dynamical Systems*, 60, (Springer-Verlag, New York, 1991).

[17] R. Brown, N. F. Rulkov and N. B. Tufillaro, Physical Review **E 50**, 4488-4508 (1994).

[18] O. E. Rössler, Physics Letters A **71**, 155 (1979).

[19] We have found that multidimensional minimization algorithms found in W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes*, (Cambridge University Press, New York, 1986), such as *amoeba* and *Powell's method* perform the desired task adequately.

[20] W. L. Brogan, *Modern Control Theory* , 359 (Prentice Hall, Upper Saddle River, New Jersey, 1991).