**Approximating chaotic time series through unstable periodic orbits**

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There are many noise reduction methods for chaotic signals, but most only work over a limited signal to noise range. If chaotic signals are to be used for communications, noise reduction techniques which can handle larger amounts of noise (or deterministic noise) are needed. I describe here a method of approximating a chaotic signal by constructing possible sequences based on unstable periodic orbits. The approximation is good enough to distinguish between chaotic attractors, even when large amounts of noise are added to the chaotic signal.

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## Introduction

There has been much work published on removing noise from chaotic signals [1-7]. Much of this work was based on embedding the chaotic signal in a phase space in order to eliminate noise. In general, phase-space based noise reduction techniques are only good if one can embed the chaotic signal in phase space so that the a noise-corrupted point in the phase space is not too far from its noise-free location. The noise reduction techniques considered were usually limited to noise on the order of 10% of the amplitude of the chaotic signal, although some techniques could handle noise as large as the chaotic signal.

At the same time, chaos is being considered as a communications signal [8-18]. In many situations, communications signals are subject to large amounts of additive noise. There are spread-spectrum communications systems that can function when the noise is 1000 times as large as the signal [19]. "Noise" may include random noise, other chaotic signals used as carriers by other transmitters, and multipath interference, which includes delayed versions of the same chaotic signal. If chaotic signals are to be used for communications, then noise reduction techniques are necessary that work when the signal to noise ratio is much less than 1 or the noise is deterministic.

Previous noise reduction techniques have focused on recovering an exact copy of the original chaotic signal. For some applications, it might be useful just to approximate the original chaotic signal, as long as the approximation recovered some useful property of the chaotic signal, such as which attractor the chaotic system was in. In this paper, I use unstable periodic orbits to approximate a chaotic signal. Although there are an infinite number of unstable periodic orbits in a chaotic attractor, many properties of the chaotic attractor may be recovered from only the lowest period orbits [20-26]. It is possible to construct an approximate skeleton of the chaotic attractor by stringing together unstable periodic orbits. The skeleton will not be exact because the chaotic system may be on some orbits only for a short time, but the skeleton may be good enough for some purposes. This procedure will work better if the chaotic system stays near each unstable periodic orbit long enough that it completes a full cycle of the orbit, making it easier to identify the orbit.

**Basic method**

The basic procedure I use is as follows: 1) Extract low period unstable periodic orbits from the chaotic system. Any of the known methods for extracting unstable periodic orbits will work for this [21, 25, 27]. I usually use the method of close returns applied to a chaotic time series [25]. If an unstable period orbit is so unstable that it does not show up in the time series, then it will not contribute to making a good approximation to the signal. Fig. 1, for example, shows the first 5 unstable periodic orbits for the Lorenz system of eq. (1).

2) Piece together individual unstable periodic orbits to create longer periodic orbit sequences. First, one must decide how to match up unstable periodic orbits, i. e. how could one orbit lead into the next. The most general way to do this would be first to choose a point on orbit A to be the final point on that orbit. The final point on orbit A, call it $\boldsymbol{x_0}$ , is a new initial condition. Using knowledge of the dynamics, it should be possible to predict the future trajectory from point $\boldsymbol{x_0}$. Next, assume some small error in $\boldsymbol{x_0}$. This small error can lead to a range of possible values for the point that comes one timestep after $\boldsymbol{x_0}$. The initial point on orbit B, the unstable periodic orbit that follows orbit A, should be within this range. I use this prediction method below with the logistic map, although in practice a simpler method may be used with flows. I describe the simpler method in the section below on the Lorenz system.

Some (or many) of these periodic orbit sequences may not actually show up in a time series signal. One may compare (as shown below) the periodic orbit sequences with many time series signals to see which sequences are actually present as good approximations to the time series. Periodic orbit sequences that are never useful as approximations to the time series may be eliminated from consideration.

3) Take all periodic orbit sequences of a given length. Compare all of these sequences with an equal length piece of the chaotic time series. I use a cross-correlation to compare. I first subtract the mean value from the time series signal and from each individual periodic orbit sequence, so that all signals are zero mean. I calculate the cross-correlation between each sequence and the piece (or segment) of the time series. The cross-correlation is normalized so that the largest possible value is 1, which occurs for identical signals. I take the sequence with the largest cross-correlation to be the best approximation to the piece of the chaotic time series. I then repeat this procedure for the next segment of the chaotic time series. It is necessary to match the phase of the periodic orbit sequence to the phase of the time series segment, but this is easily done by using FFT techniques to compute the cross-correlation [28].

Cross-correlation is commonly used in standard spread-spectrum systems [19, 29]. To be rigorous, cross-correlation techniques are used with a set of orthogonal time series, so that the cross correlation between the time series is zero. In the examples presented in this paper, the signals are not actually orthogonal. The cross-correlation between two signals is not zero for these short sequences. Nevertheless, the cross correlation between two identical periodic orbit sequences will be one. Because a chaotic system has a one or more positive Lyapunov exponents, the cross correlation between non-identical sequences will be less than one, so it is possible to distinguish between periodic orbit sequences using cross-correlation. The accuracy of the cross-correlation calculation will increase as the sequence length increases and decrease as the noise level increases.

There is a trade-off in extracting a signal from noise: the longer the sequences, the more of them there are to be compared. The number of sequences should increase roughly exponentially with their length. I show below that not all periodic orbit sequences that I construct actually occur in the attractor, so it should be possible to limit the increase in the number of sequences that need to be considered. It may be that work on classifying chaotic attractors using grammars [30] or templates [31] may help limit the number of sequences under consideration.

**Lorenz system**

I start with the Lorenz system as an example. The Lorenz equations I use here are

$$\begin{aligned}\frac{dx}{dt} &= 16(y-x)\\ \frac{dy}{dt} &= -xz+45.92x-y\\ \frac{dz}{dt} &= xy-4z\end{aligned} \qquad (1).$$

The equations were integrated with a 4-th order Runge-Kutta integration routine with a time step of 0.028.

I found the periodic orbits for the Lorenz system up to period 4 using a Newton-Raphson algorithm [32]. Figure 1 shows an x-y projection for each of these orbits. As can be seen, there are only a few basic types of motion for the Lorenz system. We see motion about one center, motion about both centers, or combinations. Motion around one center or around 2 centers occurs at incommensurate frequencies, so these orbits are not true period 1,2,3 and 4 orbits, but I will label them as such for convenience. The length of the period 2 orbit is close to twice the length of the period 1 orbit, the length of the period 3 orbit is close to 3 times the length of the period 1 orbit, and so on.

As an example of the lack of cross-correlation between different periodic orbits, I constructed sequences consisting only of the *x* component of the period 2 and the period 3 orbits. I shifted the phase of one of the sequences to find the maximum of the cross-correlation function between the two sequences. When the two sequences were 4 cycles long, the maximum of the cross-correlation was 0.49. When the two sequences were 8 cycles long, the maximum of the cross-correlation was 0.24.

The unstable periodic orbits for the Lorenz system may be combined into periodic orbit sequences. In order to combine the orbits into sequences using the prediction method described above, it is necessary to know *x*, *y* and *z* points at the end of each unstable periodic orbit. Because the Lorenz system is a flow, however, there is a simpler approximate method to combine unstable periodic orbits into sequences. A flow system changes continuously from one time step to the next, so one may attempt to combine orbits by making the sequence roughly continuous from the end of one unstable periodic orbit to the beginning of the next. One may choose a particular arbitrary value of the signal to match orbit A and orbit B(in a periodic system, this would be the same as matching the phases of the two orbits). For example, one may use the point where the orbit crosses zero going in the positive direction as the matching point. If the orbit does not cross zero, one may use the point where the orbit most closely approaches zero. Figure 2 shows an example of a sequence consisting of a period 1, period 2 and period 1 orbit, all from the *x* variable of the Lorenz system. The arbitrary matching condition does lead to some glitches, as can be seen in Fig. 2, but the resulting sequence is good enough for an approximation.

All possible periodic orbit sequences up to a certain length are constructed. The dynamics of the particular dynamical system may limit which periodic orbit sequences are possible. In the Lorenz system, for example, the *x* variable is symmetric about zero. For any periodic orbit, both the orbit and its inverse may be used to construct a periodic orbit sequence. The period 1 orbit, however, does not cross the origin, as can be seen in Fig. 1. One could not have a sequence consisting of the period 1 orbit followed by its inverse because the Lorenz system cannot cross zero while in a period 1 orbit. It might be possible, on the other hand, to have a period 1 orbit, a period 2 orbit, and then the inverse of the period 1 orbit.

Some orbits have several zero crossing points, so there are several possible phases in which the orbit may enter into a periodic sequence. The period 3 orbit, for example, has 2 possible phases. Periodic orbit sequences of a given length are constructed by combining all unstable periodic orbits and their inverses in all possible phases, eliminating combinations obviously not allowed by the dynamics. For the Lorenz system, there were 49 possible sequences of length 4, such as period 1+ period 3, period 2+period 1 + period 1, a single period 4, etc. The 49 possible period 4 sequences were then combined to produce 2401 sequences of length 8.

I then calculated the cross correlation between each periodic orbit sequence and an equal length segment from a Lorenz *x* time series, subtracting the mean, normalizing and checking for the proper phase as above. I took the sequence that yielded the largest product to be the best approximation to that segment of the Lorenz *x* time series.

Figure 3 shows the results of fitting sequences to a Lorenz time series from the *x* variable. Figure 3(a) is the original Lorenz time series before the mean has been subtracted. Figure 3(b) is an approximation using sequences of length 8 (the mean value has been added back in for the figure). Figure 3(c) is an approximation when a Gaussian white noise signal with an RMS value of 20 has been added to the original Lorenz signal. The RMS of the Lorenz signal is 12.7, so the RMS signal to noise ratio was near 0.5, but the approximation was still good.

In an attempt to reduce the computational burden, I took 1000 period 4 segments from a Lorenz *x* time series from eq. (1) and fit each one with one of the 49 possible period 4 sequences. Some of the period 4 sequences were not used, so I eliminated these sequences. I then built period 8 sequences by combining the remaining period 4 sequences. As a result, I was left with 961 period 8 sequences, a reduction by a factor of almost 3 in the total number of sequences that I needed to consider. One might also consider eliminating sequences that are not used very often. The quality of the approximation might suffer, but the computation time could be reduced.

**Circuit data**

I used data from an electronic circuit to see if this approximation technique could work with real data. The circuit was similar to a circuit described in [33]. The circuit was described by the equations:

$$\frac{dx}{dt} = -\alpha\left[1.47x - \beta y + z - w - g_1(w)\right]$$
$$\frac{dy}{dt} = -10\alpha\left[x + 0.2y\right]$$
$$\frac{dz}{dt} = -4.5\alpha\left[y + 0.2z\right]$$
$$\frac{dw}{dt} = -10\alpha\left[-2.49x + 0.5w + g_2(w)\right]$$

$$g_1(w) = \begin{Bmatrix} -\mu & w \le -\mu \\ w & -\mu < w < \mu \\ \mu & w \ge \mu \end{Bmatrix}$$
$$g_2(w) = \begin{Bmatrix} m_2 w - 2(m_1 - m_2) & w \le -2 \\ m_1 w & -2 < w < 2 \\ m_2 w + 2(m_1 - m_2) & w > 2 \end{Bmatrix} \tag{2}$$

with $\alpha = 10^4$, $\beta = 1.88$, $\mu = 0.635$, $m_1 = -0.5$, and $m_2 = 0.5$, Figure 4(a) shows a time series of the *w* signal from the circuit, digitized at 100,000 points/sec.

I digitized all 4 signals from the circuit and used the method of close approaches to find periodic orbits up to period 4. As with the Lorenz system, I then constructed all possible sequences up to length 8 using the *w* signal. For this system, there were 768 such sequences. There are fewer sequences than for the Lorenz system because the circuit must encounter a period 4 orbit for the *w* signal to cross zero. In the Lorenz system, the *x* signal could cross zero on a period 2 orbit.

Figure 4(a) shows the *w* time series from the circuit. Figure 4(b) shows the approximation to the circuit *w* time series. There are places where the approximation good and places where the approximation is not as good. Between 8 and 10 ms, it appears that the circuit spirals in to an unstable fixed point and then shoots out. This motion does not stay near any single low period unstable orbit, so it is not well approximated.

Figure 4(c) shows the approximation to the time series when Gaussian white noise with an RMS amplitude of 4 has been added to the original signal. The RMS amplitude of the $w$ signal from the circuit is 1.99, so the signal to noise ratio is 0.5. The approximation is not as good with the added noise, but the approximate signal is still usually on the same side of zero as the original signal.

**Sending a signal**

If the approximation to the chaotic signal is to be useful, it must recover some property of the chaotic signal that we can control. One property that the approximation can pick up is which attractor the chaotic system is in. As a communications example, I send either a signal from an asymmetric chaotic attractor or its inverse. Because the attractor is not symmetric about zero, I can tell by fitting periodic orbits whether I am sending the unaltered chaotic signal or its inverse.

For my test system, I used eq. (2) with $\alpha = 1$, $\beta = 2.4$ and $\mu = 0.4$. Eq. (2) was integrated with a 4-th order Runge-Kutta integration routine with a time step of 0.02 . Periodic orbits were found for the resulting attractor by using the method of close approach. Eq. (2) was then integrated with a time step of 0.08 to produce a time series of the $x$ signal. The average value of the $x$ signal (0.028) was subtracted from the $x$ signal to produce $x_a$ , which had zero mean. Sequences of length 8 periods based on unstable periodic orbits up to period 4 were constructed as before, and their mean values were subtracted. Figure 5 shows the zero mean $x$ signal, $x_a$.

To encode a signal, $x_a$ was multiplied by $s = \pm 1$. The sign of $s$ was switched every 800 points, or about 26 cycles, to produce a communications signal $x_c = sx_a$ . The periodic orbit sequences and their inverses were fit to the time series as before. To recover the signal $s$ , I recorded whether the sequence that best fit each segment of the time series was inverted or not. Figure 6(a) shows the value of $s$ , while Fig. 6(b) shows $s$ recovered from the periodic orbit approximation to the time series. The magnitude of $s$ shown in Fig. 6(b) is the value of the cross correlation between the segment of the time series and the periodic orbit sequence that best fit it.

Noise does not destroy the ability to communicate. Figure 6(c) shows the recovered value of $s$ when Gaussian white noise with an RMS amplitude twice the RMS amplitude of the corrected $x$ signal was added to the $x$ signal. The $s$ signal is still recovered accurately. Using longer sequences should allow signal recovery at lower signal to noise ratios, although the computational burden will increase.

**Other kinds of interference**

Multipath interference can also degrade communications signals. Multipath interference occurs when the communications signal is reflected from buildings or other objects. The reflections arrive at the receiver at a later time because they travel a different path. If the time delay is a half integral number of periods of the carrier signal, the interference can cancel out most of the received signal. The interfering signal is also difficult to separate from the received signal because it is at the same frequency.

As a test of multipath interference, I delayed and attenuated the communications signal $x_c$ to produce the signal $Ax_c(\tau)$, where $A$ is an amplitude multiplier and $\tau$ was the delay time. I added these signals together to produce $x_c + Ax_c(\tau)$.

I used delays of 1/2 data period (800 points, about 13 cycles) and 1/2 cycle (about 60 points). In both cases, I could recover the information signal $s$ for amplitude multipliers $A$ up to 0.8.

I also tested the effect of periodic interference on the communications signal. The power spectrum of the $x$ signal from eq. (2) has a large peak at about 0.42 Hz, corresponding approximately to the frequency of 1 cycle. I added a sine wave at this frequency to the communications signal $x_c$ . Again I used periodic orbit sequences of length 8 to recover the information signal $s$ . I was able to recover the information signal when the sine wave RMS amplitude was up to about 3/4 of the RMS amplitude of the communications signal. Using longer sequences should allow for recovery at lower signal to noise ratios.

**Maps**

It is natural to attempt this sort of approximation in maps. In a map, a period 2 orbit is really 2 steps long, not 30 steps as in our last example, so computation time should be speeded up. I used the logistic map:

$$x_{n+1} = 4x_n\left(1 - x_n\right) \tag{3}$$

which I iterated with 64 bit precision. I then found periodic orbits up to period 8 for the map.

I created longer sequences from orbits up to period 8. I used all possible phases of each orbit in making my sequences. The prediction method described above was used to determine which orbits could follow each other. At the end of each orbit, I extrapolated the next point on the orbit using eq. (3). In constructing sequences, I followed each orbit only with another orbit whose first point was within some tolerance of the extrapolated point from the previous orbit. By observing the behavior of the map, I set this tolerance at 0.1.

All sequences of length 8 were combined to create sequences of length 16. I found 18,601 such sequences. In order to lessen the computational time, I then checked which sequences of length 8 were actually likely to show up in the map. The map was iterated for 10,000 segments of length 8, and I recorded which orbit sequence was the best fit. As before, the mean was subtracted from each periodic orbit sequence before the cross-correlation calculation. Many periodic orbit sequences were never the best fit, so they could be eliminated from consideration. I then combined this reduced set of length 8 sequences into sequences of length 16. There were now only 650 sequences of length 16, a reduction by a factor of 28 in the number of sequences needed. Figure 7(a) shows a time series from the map of eq. (3), while Fig. 7(b) shows an approximation to that time series using unstable periodic orbits (the mean value has been restored for the figure).

The logistic map also worked in a simple communications scheme. I first subtracted the average value from a time series from the logistic map to produce a time series with zero mean. I then multiplied the zero mean time series by an information signal $s = \pm 1$ to produce a communications signal. The sign of $s$ flipped every 48 map iterations. The communications signal was then approximated with periodic orbit sequences of length 16. The approximation revealed whether the communications signal was inverted or not. The sign of $s$ was successfully recovered when I added to the communications signal a Gaussian white noise signal with an RMS amplitude half that of the communications signal. It should be possible to recover $s$ from higher noise levels by using longer periodic orbit sequences, although there will then be more sequences to compare to.

**Conclusions**

I have described communications systems with two symbols, ±1. It should be possible to create signals with more symbols by using more different attractors. Even with large noise levels, the periodic orbit approximation can distinguish between different chaotic attractors. The periodic orbit approximation technique should be useful even when there are other chaotic signals interfering with the desired signal.

The weakest point of using periodic orbit sequences to approximate chaotic signals is the amount of computation needed. By using long enough sequences, it is in principle possible to extract a chaotic communications signal from very large amounts of noise, but the number of periodic orbit sequences increases exponentially with sequence length. It may be possible to limit the number of sequences by eliminating periodic orbit sequences that do not naturally occur, or by using the communication scheme of Hayes [13], where the transmitter is controlled to produce specific sequences of unstable periodic orbits.

Improved techniques for searching for the best fit among the periodic orbit sequences could also speed up the calculation. It should be possible to calculate cross-correlations between all periodic orbit sequences. The periodic orbit sequences could then be grouped in a tree structure according to how closely they were correlated with each other. Searching for the best fit periodic orbit sequence would then be a matter of searching through the tree, which should be considerable faster than comparing to every orbit. Another possible improvement is to check the most often used periodic obits sequences first, and use the first sequence for which the cross-correlation exceeds some threshold.

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**Figure Captions**

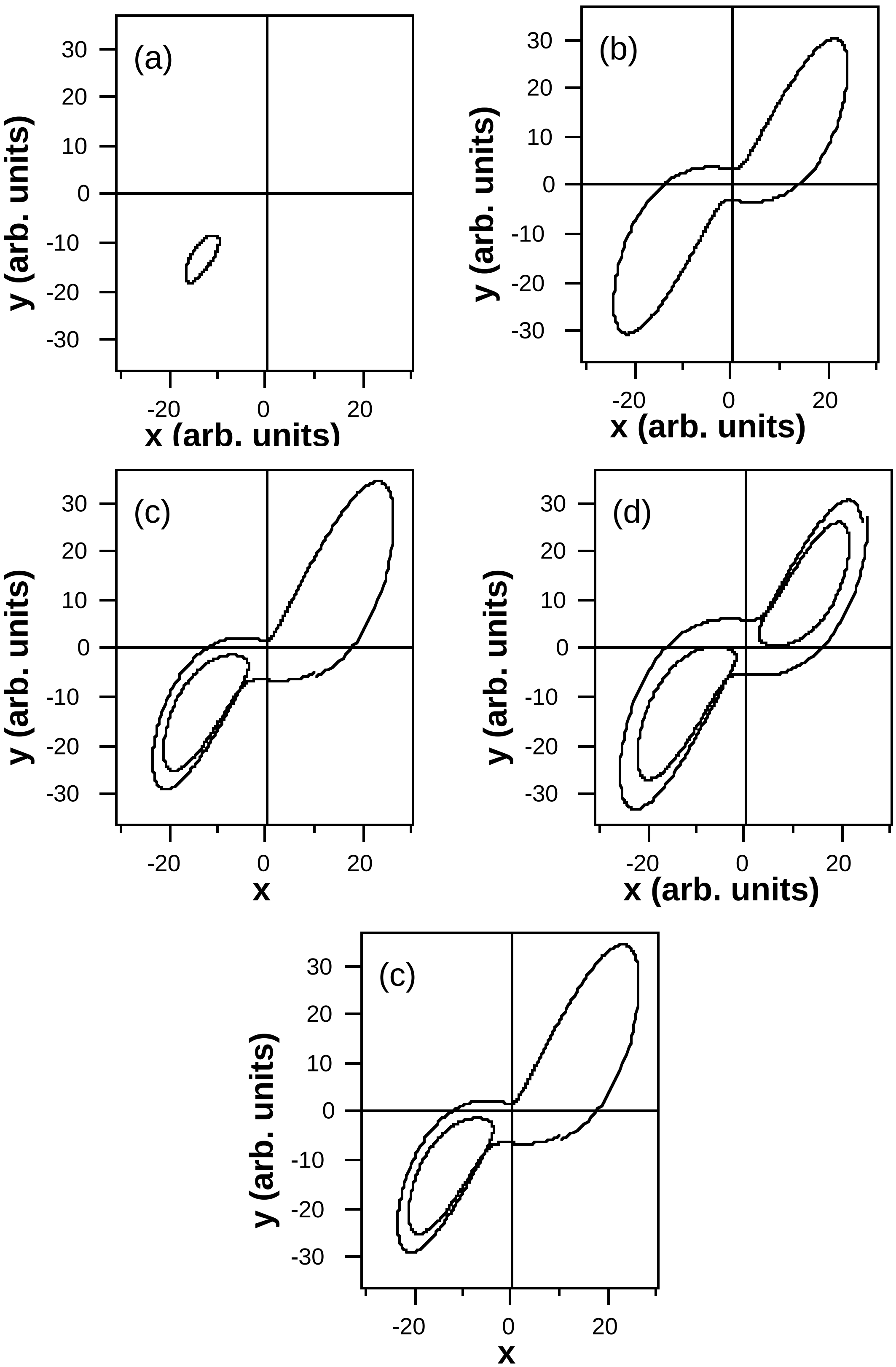

Fig. 1. Unstable periodic orbits for the Lorenz system of eq. (1). (a) is period 1, (b) is period 2, (c) is period 3, (d) and (e) are period 4.

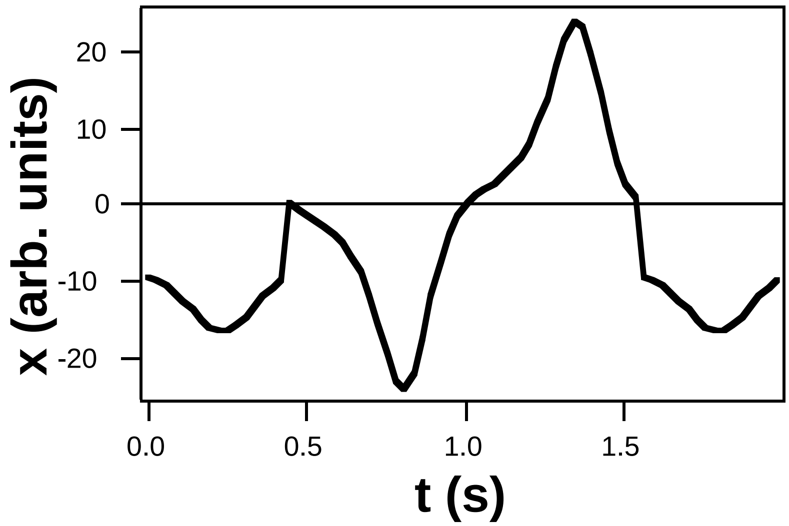


Fig. 2. Sample of a periodic orbit sequence for the Lorenz system of eq. (1). The sequence consists of a period 1 orbit followed by a period 2 orbit and then another period 1 orbit.

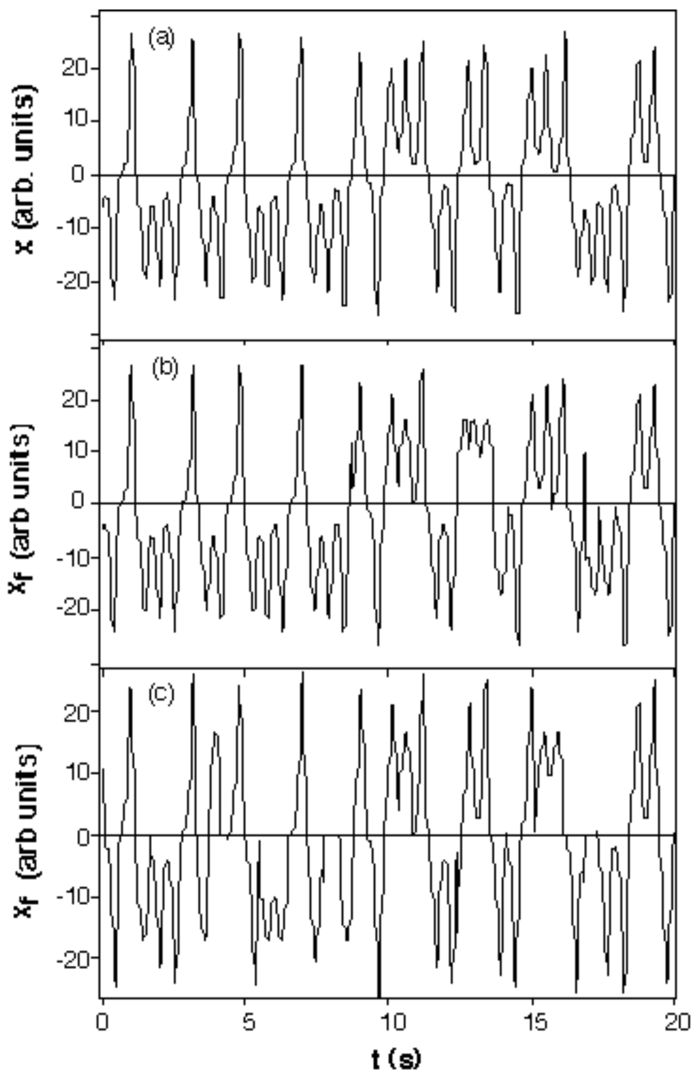


Fig. 3. (a) is the $x$ signal from eq. (1), (b) is $x_f$ an approximation to the $x$ signal constructed from sequences of unstable periodic orbits, and (c) is the same approximation when Gaussian white noise twice as large as the $x$ signal has been added to the $x$ signal. While the approximation in (c) does not look that good, it is usually on the same side of zero as the time series in (a), so some topological properties are still captured.

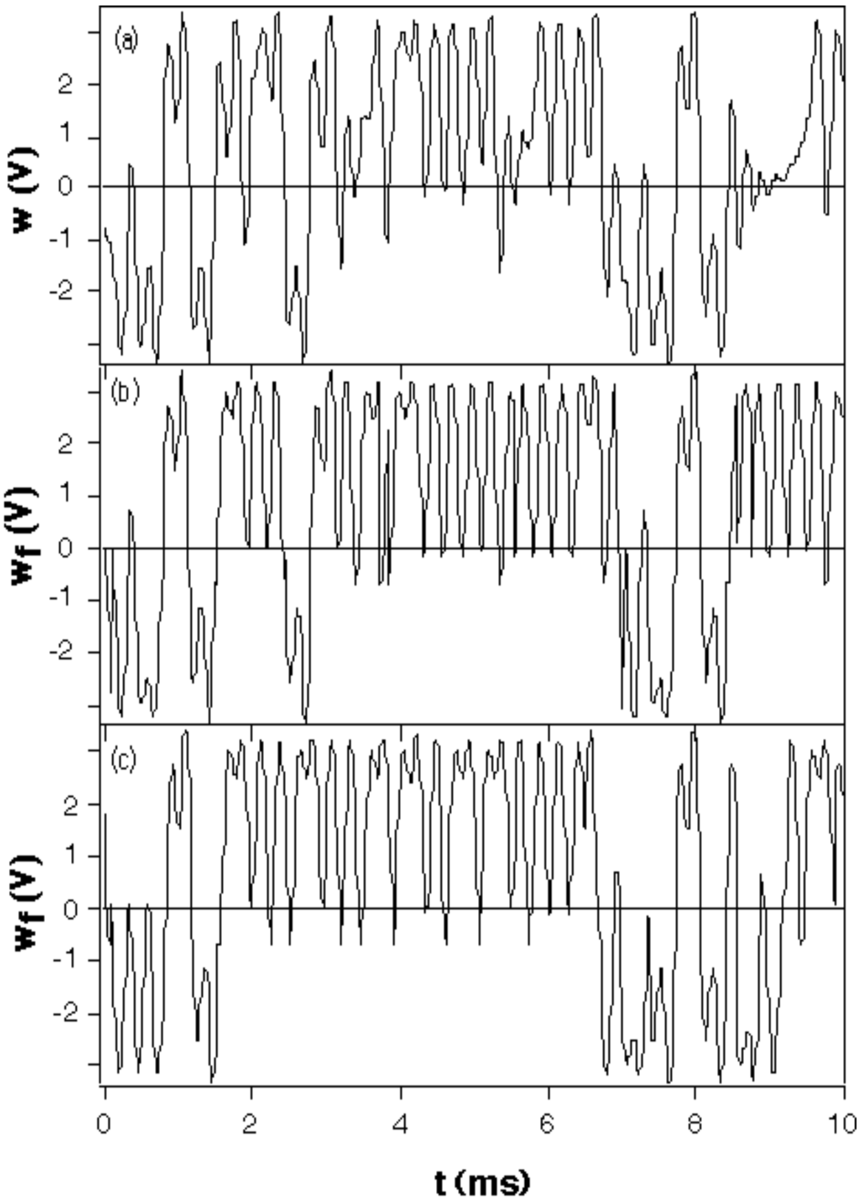


Fig. 4. (a) is a signal *w* from a 4-d circuit described in eq. (2), (b) is an approximation $w_f$ to the *w* signal, constructed from sequences of unstable periodic orbits, and (c) is the same approximation when Gaussian white noise twice as large as the *w* signal has been added to the *w* signal. Note the poor approximation between t=8 and 10 ms, when the circuit is not near an unstable periodic orbit.

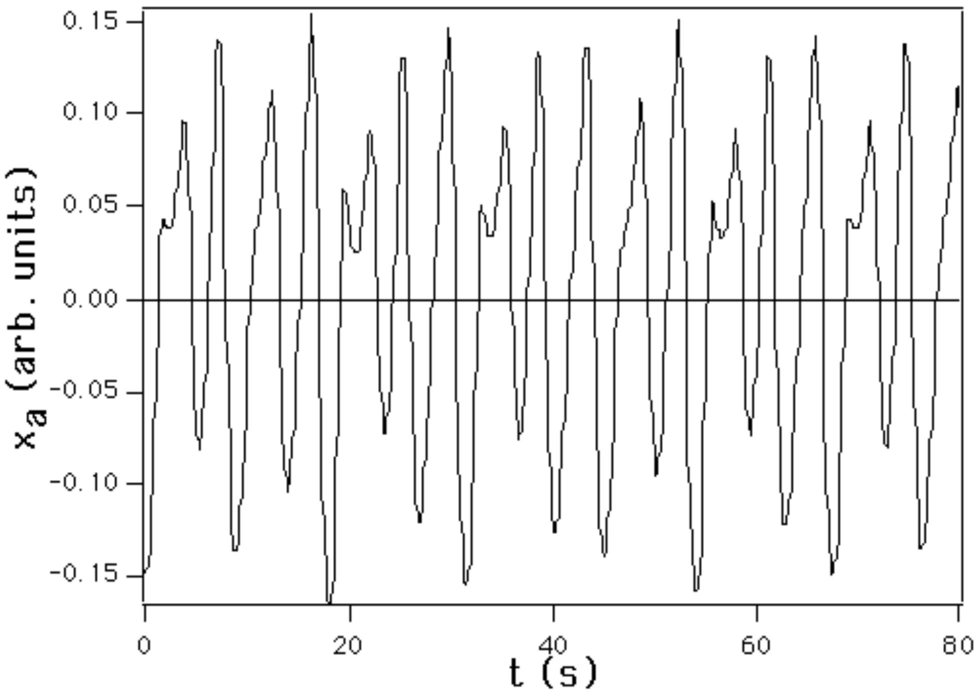


Fig. 5. Zero mean signal $x_a$ from the a simulation of eq. (2) with $\alpha = 1$, $\beta = 2.4$ and $\mu = 0.4$.

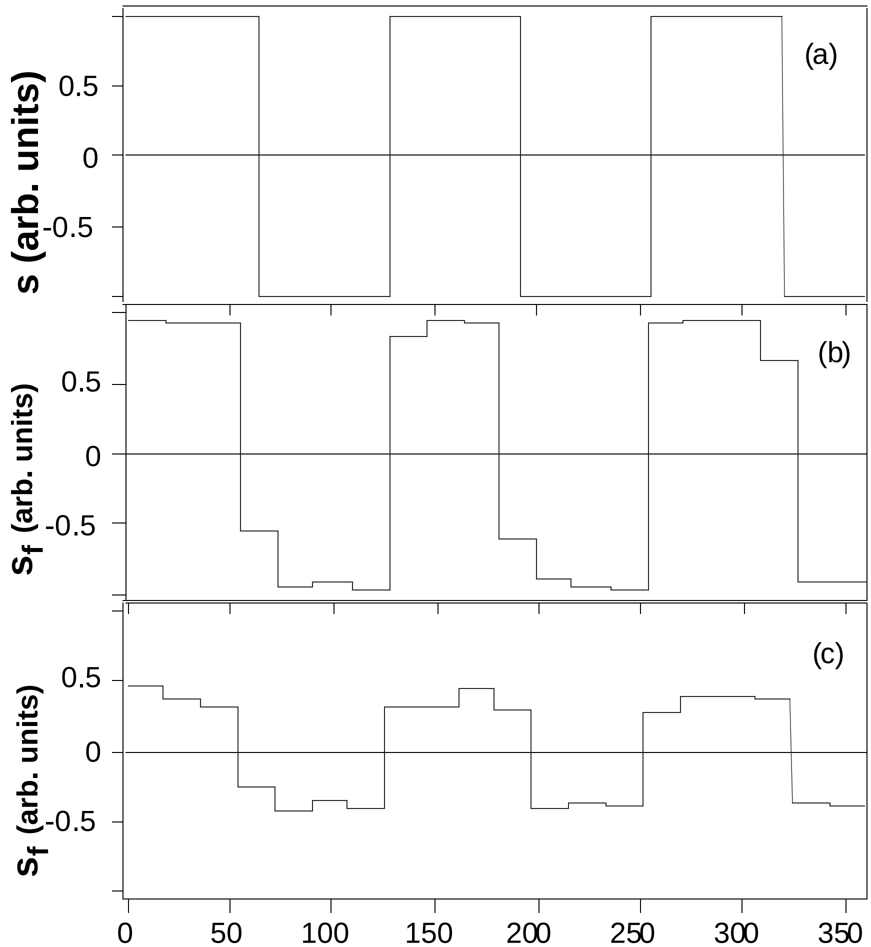


Fig. 6. (a) is a communications signal $s$ used to modulate the carrier signal of Fig. 4. (b) is the recovered modulation signal $s_f$ . The magnitude of $x_f$ is the cross correlation between the modulated communications signal $x_c$ and its periodic orbit approximation. (c) is the recovered modulation signal $s_f$ when a Gaussian white noise signal twice as large as the communications signal $x_c$ has been added to the communications signal.

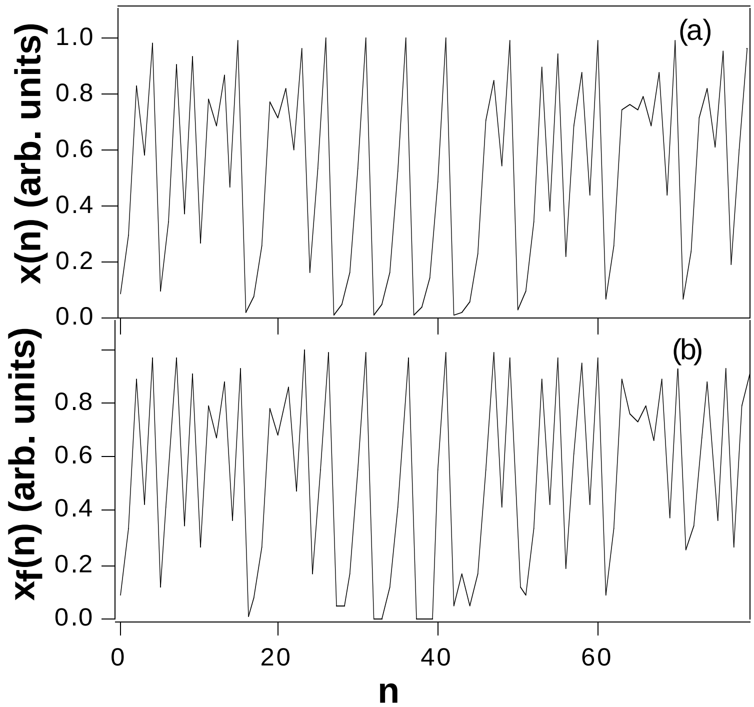


Fig. 7. (a) is the signal *x(n)* from the logistic map of eq. 3. (b) is an approximation to the logistic map signal using periodic orbit sequences of length 16.