Noise-Resistant Chaotic Maps

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Synchronized chaotic systems are highly vulnerable to noise added to the synchronizing signal. It was previously shown that chaotic circuits could be built that were less sensitive to this type of noise. In this work, simple chaotic maps are demonstrated that are also less sensitive to added noise. These maps are based on coupling a shift map to a digital filter. These maps are simple enough that they should help lead to an understanding of how noise-robust chaotic systems work.

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The positive Lyapunov exponents associated with chaotic systems mean that a signal from one isolated chaotic system will not synchronize with a signal from any other chaotic system. Chaotic systems may be synchronized by coupling them together, and these synchronized chaotic systems may have applications in the field of "spread spectrum" communications, where broad carrier signals are used to prevent interference with other signals or conceal the presence of the signal itself or an encoded message. One major problem in applying synchronized chaotic systems to communications is that if the signal that couples the chaotic signals is transmitted (over a radio link for example), then other signals from other transmitters or background noise are also present, and these other signals are picked up be the receiver along with the coupling signal. These other signals add to the coupling signal and get injected into the chaotic system, where they ruin the synchronization. For existing synchronized chaotic systems, if the added signals approach the size of the coupling signal, then synchronization is degraded to the point where no communication is possible

This paper demonstrates that it is possible to build simple chaotic maps that can synchronize arbitrarily closely even when large interfering signals have been added to the synchronizing signal. These simple maps consist of two parts: a modulus map, where a number is multiplied by a constant and only the fractional part is retained, and a digital filter, which is a map that is designed to have a certain frequency response. Gaussian white noise is used to approximate the interfering signals. While the maps in this paper are not practical for a communications system, in previous work analog electronic circuits were built that synchronized in the

presence of large noise. This paper uses simple maps to aid in understanding how the noise-resistant chaotic systems work.

Introduction

Synchronized chaotic systems have been suggested for applications in communications[1-9], but in truth, their high noise sensitivity makes the application of these systems unlikely. In recent work with an analog electronic circuit, it was shown that it was possible to build synchronized chaotic systems that can achieve arbitrarily small synchronization error even when large amounts of noise are added to the transmitted signal[10]. As of yet, no good theory exists to explain the noise reduction effect in these circuits. I present here some simple map examples where synchronization error in a noisy environment is reduced. Because of the particular map design, the practical application of these maps is limited, but their simplicity should make it easier to understand the noise-reduction effect

Simple maps

The first example uses a coupled pair of 1-d maps, where one of the maps consists of a multiplication by a constant and a modulus, while the other map acts as a low pass filter. The coupled maps are given by

$$\begin{aligned} x_1(n+1) &= 1.5x_1(n) - 0.1\tau x_2(n) \mod 1 \\ x_2(n+1) &= \left(\frac{4}{\tau}\right)x_1(n) + \left(1 - \frac{1}{\tau}\right)x_2(n) \end{aligned} \qquad (1).$$

Looking at the x_2 map alone, it can be seen that its slope will approach 1 as τ increases ,

while the term driving the map will decrease, so as τ increases the x_2 part of the map will

change more and more slowly in response to the x_1 driving term, acting as a low-pass filter. As long as $\tau > 1$ and $< \infty$, the x_2 map acts as a low pass filter. From digital filter theory [11], the absolute magnitude of the gain of the x_2 map is $A(f) = 1/\sqrt{1+\beta^2 - 2\beta\cos(2\pi f)}$ where f is the frequency (ranging from 0 to 0.5) and $\beta = (1-1/\tau)$. For $\tau = 1$ ($\beta = 0$) the magnitude of the gain is 1.0 for all frequencies, while $A(f)$ decreases with f for $\tau > 1$. As τ becomes very large, β approaches a limiting value of 1.0, causing the cutoff frequency f_c (defined as the frequency where the amplitude response is down by a factor of 2 from its maximum) to approach a limiting value, meaning that little additional filtering effect is gained for values of $\tau >> 10$. The factors of τ in the x_1 equation and $4/\tau$ in the x_2 equation are present simply to scale the map values into a convenient range. It should be noted that the x_2 map does not contain a modulus operator. For $\tau = 10$, the eigenvalues of the map are $1.2 \pm 0.56i$. The factor of τ in the x_1 map is used to influence the stability properties of the response system.

Figure 1 shows the power spectra of the signals from the map for $\tau = 10.0$. Figure 1(a) is a power spectrum of $x_1(n)$, while 2(b) is a power spectrum of $x_2(n)$.

The response system is

$$\begin{aligned} x_t(n) &= x_1(n) + \eta \\ y_1(n+1) &= 1.5y_1(n) - 0.1\tau y_2(n) + 1.5\left(x_t(n) - y_1(n)\right) \bmod 1 \qquad (2). \\ y_2(n+1) &= \left(\frac{4}{\tau}\right)y_1(n) + \left(1 - \frac{1}{\tau}\right)y_2(n) \end{aligned}$$

The term η represents a Gaussian white noise signal. Variations in the stability of the response system may change its synchronization behavior in the presence of noise, so this coupling configuration was chosen so that the eigenvalue of the response map with the largest absolute magnitude was independent of τ, with a value of 0.63.

When even a small amount of noise is added to the transmitted signal x_t , large errors in synchronization may result because of the sensitive nature of the modulus function. With even a very small amount of noise, the value of $y_1(n)$ might be altered by the modulus function while $x_1(n)$ has not been changed. Sterling [12] has developed a technique to correct for this problem: the feedback signal at the next iteration is computed and compared to the same signal when y_1 differs by the amount of the modulus:

$$\begin{aligned} y_f &= \min_{\{j=-1,0,1\}} \left\{ x_t(n+1) - [y_1(n+1) + j] \right\} \\ y_1(n+1) &\rightarrow y_1(n+1) + j_{\min} \end{aligned} \qquad (3).$$

The next value of $y_1(n)$ is corrected by adding the value of j that minimizes y_f .

Figure 2 shows the rms synchronization error δ when the noise signal η had a rms amplitude of 0.05 and τ was varied (the standard deviation of $x_1(n)$ was 0.28). The synchronization error δ was the rms value of $x_2(n) - y_2(n)$ divided by the rms value of $x_2(n)$. The dark circles in Fig. 2 show that the synchronization error decreases as τ increases. The synchronization error approaches a lower bound as τ increases because the filter cutoff frequency does not change much for $\tau >> 10$, as explained above. The noise level of 0.18 (noise rms/signal rms) is shown as a horizontal line in Fig. 2.

Since the y_2 map is the low pass filter in the response system, it is legitimate to ask if the y_1 part of the response system is necessary at all. As an alternative, the driving signal was input directly into the y_2 map:

$$y_2(n+1) = \left(\frac{4}{\tau}\right) x_t + \left(1 - \frac{1}{\tau}\right) y_2(n) \qquad (4).$$

This response system was stable. The synchronization error for this configuration is shown in Fig. 2 as open squares. The synchronization error in this case does not decrease as τ increases, but rather stays close to the noise level of x_t.

Map with band-pass filter

The noise-reduction effect can be larger in more complicated maps. The next example is a 2-d linear map coupled to an IIR (infinite impulse response) bandpass filter. In an IIR filter, delayed versions of the previous signal may be multiplied by constants and fed back into the filter. The bandpass filter is designed from standard digital filter design techniques.

A digital filter of order N performs the sums

$$y(t_n) = \sum_{k=0}^{N} \alpha_k x_{N-k} - \sum_{k-1}^{N} \beta_k y_{N-k} \qquad (5)$$

where y is the output signal and x is the input signal [11]. For an IIR filter, feedback is present, so some of the β 's are nonzero. Techniques for calculating the filter coefficients are well known. For a 2nd order bandpass filter with gain A_r, center frequency f_c, and quality factor $Q = f_c/\Delta f$,

$$\alpha_0 = \frac{lA_r / Q}{1 + l/Q + l^2};\ \beta_1 = \frac{2(1 - l^2)}{1 + l/Q + l^2};\ \beta_2 = \frac{1 - l/Q - l^2}{1 + l/Q + l^2} \qquad (6)$$

where l = cotangent(πf_c). The filter coefficients usually contain a factor to account for the sampling rate, but since this is a map, the sampling rate is set to 1.

The second map used as an example is

$$
\begin{aligned}
x_0(n+1) &= 1.6x_0(n) - 1.2x_1(n) - x_2(n) \quad \mathrm{mod}\ 1 \\
x_1(n+1) &= 0.5x_0(n) + 0.7x_1(n) \\
x_2(n+1) &= \alpha_0 x_0(n) - \beta_1 x_2(n) - \beta_2 x_3(n) \\
x_3(n+1) &= x_2(n) \\
x_t(n) &= k_0 x_0(n) + k_1 x_1(n) + \eta
\end{aligned} \tag{7}
$$

where α_0 , β_1 and β_2 are coefficients for a bandpass filter, as described above, and η is an additive Gaussian white noise term. Note that the modulus only applies to the equation for x_0 . The map is defined in this way to make it easier to use the method of Sterling [12] described above to account for errors in the response map caused by the modulus term. The signal that is transmitted to the response map is x_t , where k_0 and k_1 are determined by minimizing the largest Lyapunov exponent for the response map [13,14]. Figure 3(a) shows the power spectrum of the transmitted signal x_t , while Fig. 3(b) shows the power spectrum for the map signal x_2, both for a filter frequency $f_c = 0.01$ and $Q = 1$. Because of the mod 1 term, there is a DC offset in the map signals, but this DC term has been removed from the power spectrum plots for clarity.

The response system is based on a map that is identical to the drive system. The response map is described by

$$
\begin{aligned}
y_t(n) &= k_0 y_0(n) + k_1 y_1(n) \\
y_0(n+1) &= 1.6y_0(n) - 1.2y_1(n) - y_2(n) + b_0(x_t - y_t) \quad \mathrm{mod}1 \\
y_1(n+1) &= 0.5y_0(n) + 0.7y_1(n) + b_1(x_t - y_t) \\
y_2(n+1) &= \alpha_0 y_0(n) - \beta_1 y_2(n) - \beta_2 y_3(n) \\
y_3(n+1) &= y_2(n)
\end{aligned} \tag{8}
$$

The constants b_0 and b_1 are found by minimizing the largest Lyapunov exponent for the response system [13,14] (the *k's* and *b's* are found simultaneously). As before, the method of Sterling is used to correct for errors in synchronization caused by the modulus term. Unfortunately, this modulus correction method is observed to be ineffective when the additive noise is much greater than 10% of the transmitted signal, so only low noise simulations are possible here.

Figure 4 shows the synchronization error when Gaussian white noise was added to the transmitted signal. The rms value of the noise was 10% of the rms value of the transmitted signal. The synchronization error δ was the rms value of $(x_2 - y_2)$ divided by the rms value of x_2. The filter center frequency ranged from $f_c = 0.001$ to $f_c = 0.01$. The dark circles in Fig. 4 show the synchronization error when the response map of eq. (8) was used. As the filter frequency f_c decreased, the synchronization error also decreased. In order to demonstrate that the error reduction was not simply due to filtering by the bandpass filter, this digital filter was also used as a response:

$$\begin{aligned} x_t(n) &= x_0(n) + \eta \\ y_2(n+1) &= \alpha_0 x_t(n) - \beta_1 y_2(n) - \beta_2 y_3(n) \\ y_3(n+1) &= y_2(n) \end{aligned} \qquad (9).$$

The response system of eq. (9) is just the bandpass filter, driven by x_0 only, with 10% Gaussian noise added. The open squares in Fig. 4 show that no reduction in synchronization error was seen when the response system was a simple bandpass filter. The observed synchronization error of 0.1 is the same as the noise level.

Conclusions

In work on an analog circuit [10], the noise reduction effect was attributed to a separation of time scales between the two parts of the circuit. It can be seen here that such a separation is not necessary: the transmitted signal is actually broad-band. It is likely that the low frequency part of the map is extracting some low-frequency statistics from the broad-band signal. It would be useful to synchronize a subsystem of one of these maps which did not contain a modulus term so that the difficulties associated with this type of nonlinearity could be avoided, but so far, the noise reduction effect has not been observed in purely linear subsystems. The noise reduction effect may be observable in other types of maps where the nonlinearity is not due to a modulus, but there may be other practical difficulties in implementing other maps (or flows).The response system should be stable even when noise much larger than the transmitted signal is present, but if the response Jacobian contains nonlinear terms, it is likely that these terms will become large for large noise. The fact that the noise reduction effect is present for broad-band signals holds some promise for signal separation if other response systems can be found.

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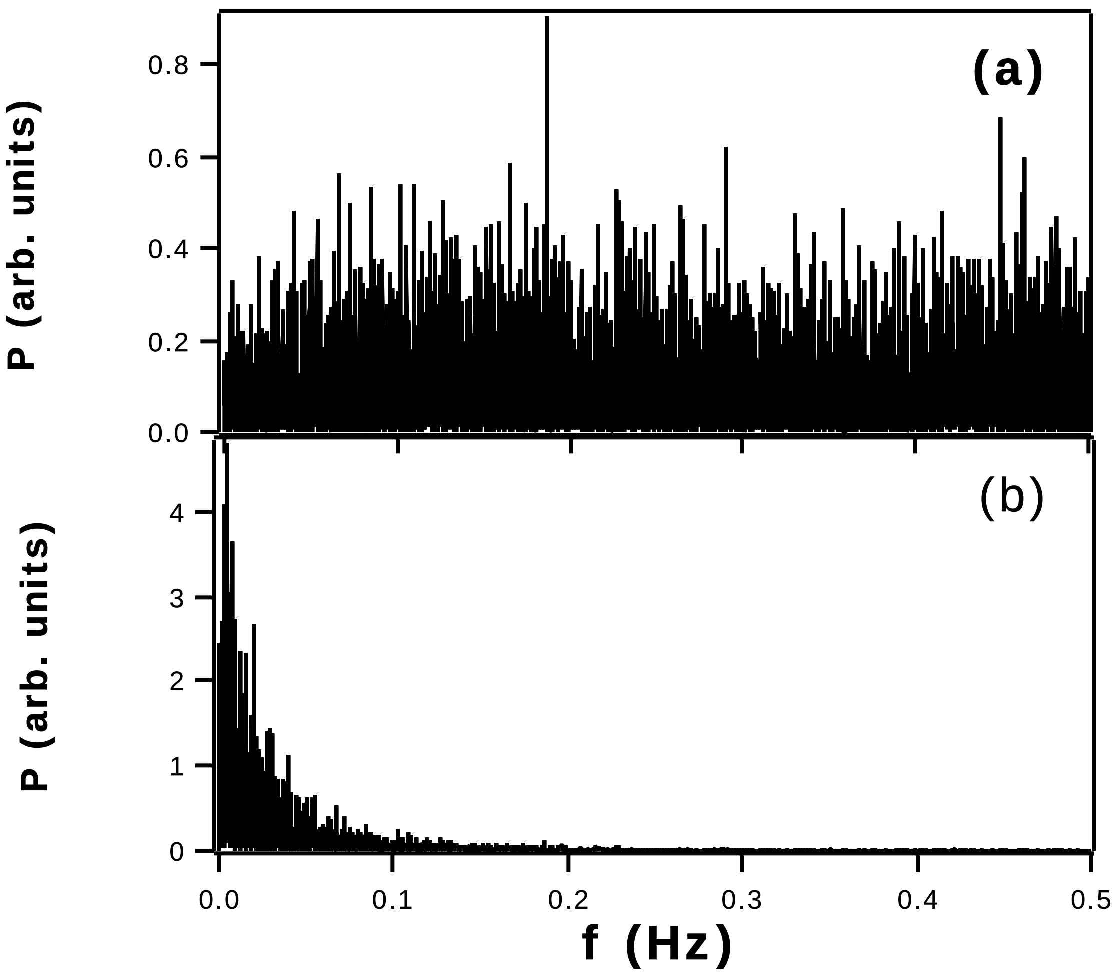

Figure Captions

Figure 1(a) Power spectrum of the transmitted signat x_t from the map of eq. (1). (b) Power spectrum of the signal x_2 from the map of eq. (1).

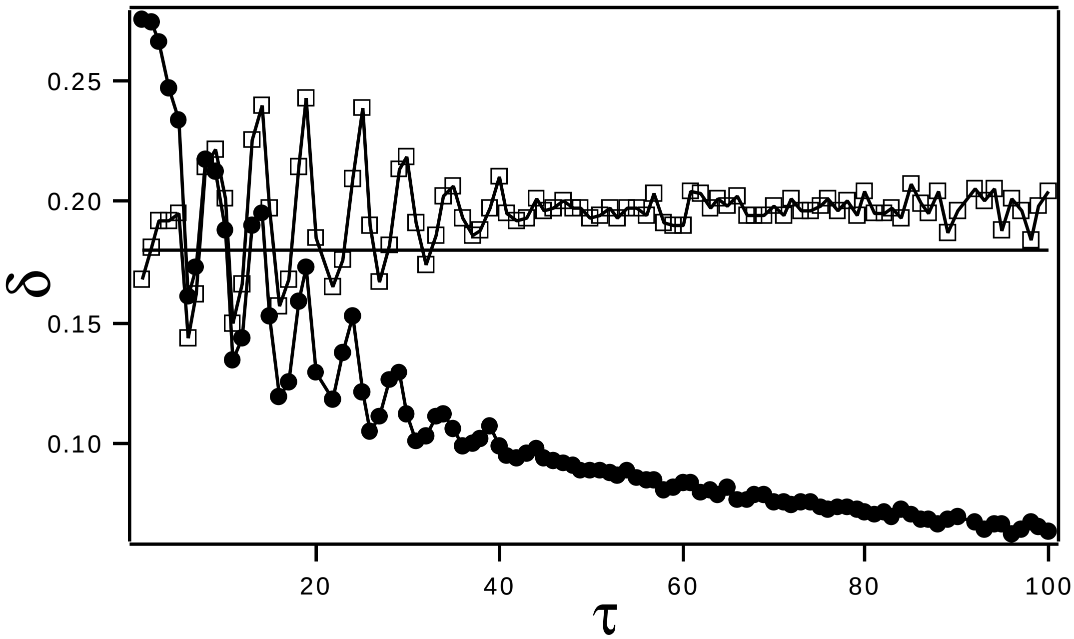

Figure 2. Dark circles show synchronization error as a function of time constant τ for the response system of eq. (2) when the added noise is about 18% of the transmitted signal. The open squares show the synchronization error when only a simple filter (as in eq. (4)) is used for the response system.

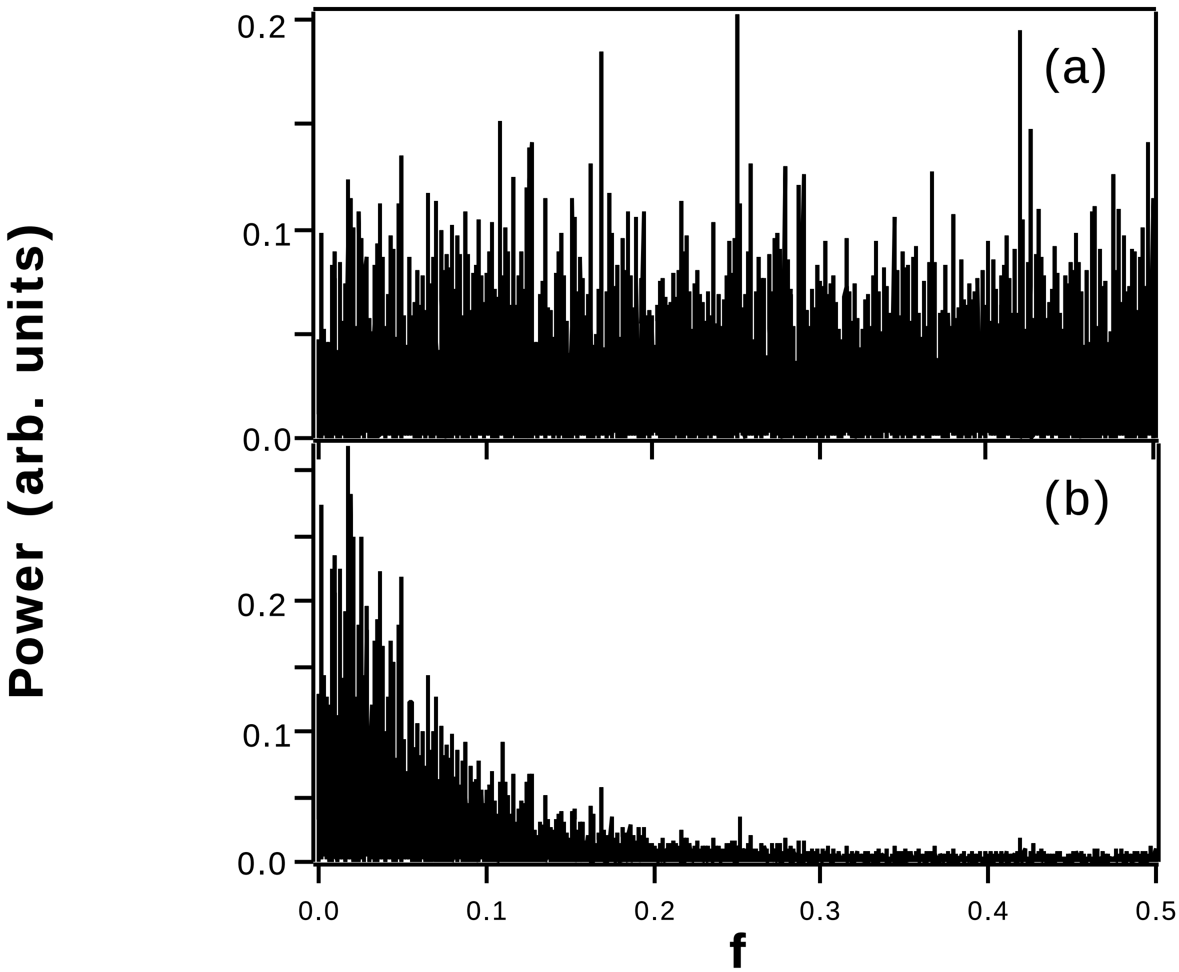

Figure 3(a). Power spectrum of the transmitted signal x_t from the map of eq. (7). (b) Power spectrum of the x_2 signal from the same map.

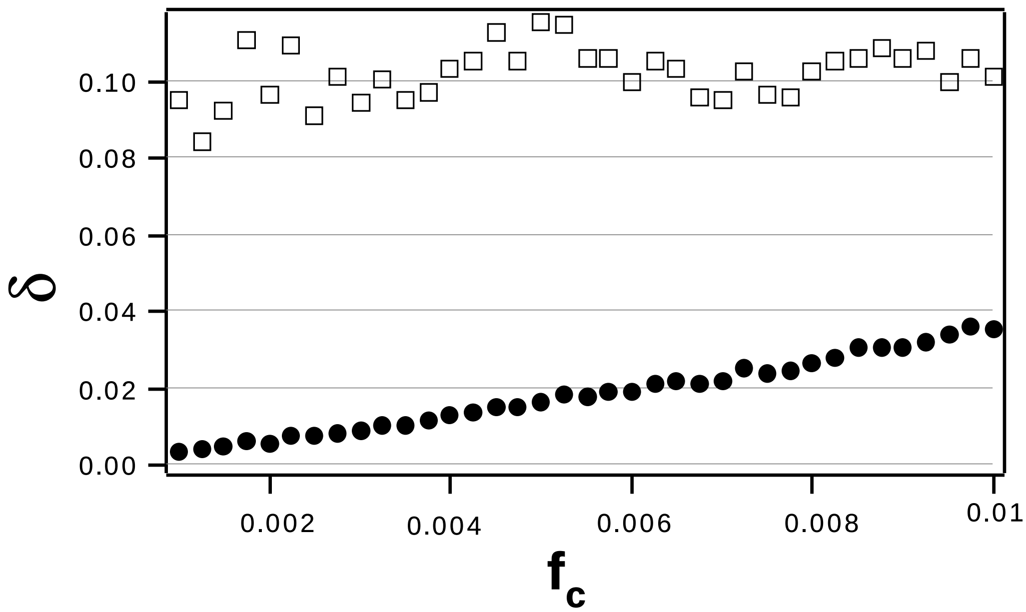

Figure 4. Dark circles show the synchronization error for the response map of eq. (8) as a function of filter center frequency f_c when the added noise was 10% of the transmitted signal. The open squares show the synchronization error when the response system was the bandpass filter of eq. (9).